E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2020 ~ Homework 11, due Thu Apr 16th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a vector space with scalar field F and $\Phi : V \times V \to F$ be a bilinear form. Identify each of the following statements as true or false:
   (a) Every $n \times n$ symmetric matrix over $\mathbb{R}$ is congruent to a diagonal matrix.
   (b) Every $n \times n$ symmetric matrix over an arbitrary field F is congruent to a diagonal matrix.
   (c) The function $Q(x, y) = xy$ on $\mathbb{R}^2$ is a quadratic form.
   (d) The function $Q(x, y) = x^2 - 4xy + xyz + z^2$ on $\mathbb{R}^3$ is a quadratic form.
   (e) The function $Q(f) = \int_0^1 x\, f(x)^2\, dx$ on $\mathbb{R}[x]$ is a quadratic form.
   (f) The function $Q(A) = \det(A)$ on $M_{2\times 2}(\mathbb{R})$ is a quadratic form.
   (g) The function $Q(A) = \det(A)$ on $M_{3\times 3}(\mathbb{R})$ is a quadratic form.
   (h) Every quadratic form over $\mathbb{R}$ is a bilinear form.
   (i) Every quadratic form over an arbitrary field is a bilinear form.
   (j) The second derivatives test will classify any critical point as a local minimum, local maximum, or saddle point.
   (k) If both eigenvalues of the 2×2 real symmetric matrix S are positive, then the graph of $(x, y) \cdot S \cdot (x, y)^T = 1$ in $\mathbb{R}^2$ will be an ellipse.
   (l) If one eigenvalue of the 2×2 real symmetric matrix S is zero and the other is nonzero, then the graph of $(x, y) \cdot S \cdot (x, y)^T = 1$ in $\mathbb{R}^2$ will be a hyperbola.

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2. For each symmetric matrix S over each given field, find an invertible matrix Q and diagonal matrix D such that $Q^T S Q = D$:
   (a) $S = \begin{bmatrix} 1 & 9 \\ 9 & 7 \end{bmatrix}$ over $\mathbb{Q}$.
   (b) $S = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & 6 \\ -2 & 6 & 7 \end{bmatrix}$ over $\mathbb{Q}$.

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3. Consider the bilinear form $\Phi[(a, b), (c, d)] = 4ac - 2ad - 2bc + 7bd$ on $\mathbb{R}^2$ and let Q be the associated quadratic form.
   (a) Write down Q explicitly and also find $[\Phi]_\beta$ for $\beta = \{(1, 0), (0, 1)\}$.
   (b) Find an orthonormal basis γ for $\mathbb{R}^2$ such that $[\Phi]_\gamma$ is diagonal, and compute the diagonalization $[\Phi]_\gamma$.
   (c) Describe the shape of the quadratic variety $Q(x, y) = 1$ in $\mathbb{R}^2$ as one of the 3 standard conic sections.
   (d) Classify the critical point of the function $Q(x, y)$ at $(0, 0)$ as a local minimum, local maximum, or saddle point.
   (e) Calculate the signature and index of Q. Is Q positive definite? Positive semidefinite? Negative definite? Negative semidefinite?

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4. Consider the quadratic form $Q(x, y, z) = 11x^2 + 40xy - 16xz - 16y^2 - 16yz + 5z^2$ on $\mathbb{R}^3$.

   (a) Find the symmetric matrix S associated to the underlying bilinear form for Q with respect to the standard basis $\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

   (b) Give an explicit orthonormal change of basis that diagonalizes Q, and find the resulting diagonalization.

   (c) Describe the shape of the quadratic variety $Q(x, y, z) = 1$ in $\mathbb{R}^3$ as one of the 9 standard quadric surfaces.

   (d) Classify the critical point of the function $Q(x, y, z)$ at $(0, 0, 0)$ as a local minimum, local maximum, or saddle point.

   (e) Calculate the signature and index of Q. Is Q positive definite? Positive semidefinite? Negative definite? Negative semidefinite?

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose that Q_1 and Q_2 are two quadratic forms on V, and let $\beta \in F$.

   (a) Show that $Q_1 + Q_2$ is a quadratic form on V. Note that $Q_1 + Q_2$ is defined pointwise, so $(Q_1 + Q_2)(\mathbf{v}) = Q_1(\mathbf{v}) + Q_2(\mathbf{v})$.

   (b) Show that βQ_1 is a quadratic form on V.

   (c) Deduce that the set of all quadratic forms on V is a vector space.

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6. Suppose $T : V \to \mathbb{R}$ is a linear operator on the real inner product space V with inner product $\langle \cdot, \cdot \rangle$. Define the map $\Phi : V \times V \to F$ by setting $\Phi(\mathbf{v}, \mathbf{w}) = \langle T(\mathbf{v}), \mathbf{w} \rangle$.

   (a) Show that Φ is a bilinear form on V.

   (b) Show that Φ is symmetric if and only if T is Hermitian.

   (c) If V is finite-dimensional, prove that Φ is an inner product on V if and only if T is a positive-definite Hermitian operator. [Hint: Show that [I3] requires all eigenvalues of T to be positive.]

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7. In multivariable calculus, the following more explicit version of the second derivatives test is often taught[1]:

   - Theorem (Second Derivatives Test in $\mathbb{R}^2$): Suppose P is a critical point of $f(x, y)$, and let D be the value of the discriminant $f_{xx}f_{yy} - f_{xy}^2$ at P. If $D > 0$ and $f_{xx} > 0$, then the critical point is a minimum. If $D > 0$ and $f_{xx} < 0$, then the critical point is a maximum. If $D < 0$, then the critical point is a saddle point. If $D = 0$, then the test is inconclusive.

   Using our general version of the second derivatives test, prove this variation. [Hint: Note that $D = \det(H) = \lambda_1\lambda_2$; then examine what information the sign of D yields about the eigenvalues λ_1, λ_2.]

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8. Let S be an $n \times n$ real symmetric matrix.

   (a) Show that S is congruent to a matrix whose diagonal entries are all in the set $\{-1, 0, 1\}$.

   (b) Prove that, up to congruence, there are exactly $\frac{1}{2}(n + 1)(n + 2)$ different real $n \times n$ symmetric matrices.

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[1] The statement of this theorem is copied directly from my multivariable calculus course notes, in fact!