E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2020 ~ Homework 10, due Thu Apr 9th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a vector space with scalar field F and $\Phi : V \times V \to F$ be a bilinear form. Identify each of the following statements as true or false:
   (a) If $V = \mathbb{R}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ is the usual inner product on $\mathbb{R}^2$, then Φ is a bilinear form on V.
   (b) If $V = \mathbb{C}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \overline{\mathbf{w}}$ is the usual inner product on $\mathbb{C}^2$, then Φ is a bilinear form on V.
   (c) If $V = \mathbb{R}$ and $\Phi(x, y) = x + 2y$, then Φ is a bilinear form on V.
   (d) If $V = F^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \det(\mathbf{v}, \mathbf{w})$, the determinant of the matrix whose columns are $\mathbf{v}$ and $\mathbf{w}$, then Φ is a bilinear form on V.
   (e) If $V = M_{n\times n}(F)$ and $\Phi(A, B) = \mathrm{tr}(AB)$, then Φ is a bilinear form on V.
   (f) If $V = M_{n\times n}(F)$ and $\Phi(A, B) = \det(AB)$, then Φ is a bilinear form on V.
   (g) If $V = C[0, 1]$ and $\Phi(f, g) = \int_0^1 x f(x) g(x)\, dx$, , then Φ is a bilinear form on V.
   (h) If $V = C[0, 1]$ and $\Phi(f, g) = \int_0^1 f'(x) g'(x)\, dx$, then Φ is a bilinear form on V.
   (i) If Φ is a symmetric bilinear form, then $[\Phi]_\beta$ is a symmetric matrix for any basis β.
   (j) If $[\Phi]_\beta$ is a symmetric matrix for some basis β, then Φ is a symmetric bilinear form.
   (k) If $\mathcal{B}(V)$ is the space of all bilinear forms on V, then $\dim_F \mathcal{B}(V) = 2n$.
   (l) Congruent matrices have the same eigenvalues.

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2. Solve the following (systems of) differential equations using any method. [Tip: the eigenvalue method is usually faster when it applies.]
   (a) Find the general solution to $\left\{ \begin{array}{rcl} y_1' & = & 7y_1 + y_2 \\ y_2' & = & 9y_1 - y_2 \end{array} \right\}$.
   (b) Find the general solution to $\left\{ \begin{array}{rcl} y_1' & = & 3y_1 - 2y_2 \\ y_2' & = & y_1 + y_2 \end{array} \right\}$.
   (c) Find the general solution to $y'' - 4y = 0$. [Hint: Set $z = y'$ and convert to a system of linear equations.]
   (d) Find the general solution to $\left\{ \begin{array}{rcl} y_1' & = & y_1 - 3y_2 + 7y_3 \\ y_2' & = & -y_1 - y_2 + y_3 \\ y_3' & = & -y_1 + y_2 - 3y_3 \end{array} \right\}$.

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3. For each bilinear form on each given vector space, compute $[\Phi]_\beta$ for the given basis β:
   (a) The pairing $\Phi((a, b, c), (d, e, f)) = ad + ae - 2be + 3cd + cf$ on $V = F^3$ with β the standard basis.
   (b) The pairing $\Phi(p, q) = p(-1)q(2)$ on $V = P_2(\mathbb{R})$ with $\beta = \{1, x, x^2, x^3\}$.
   (c) The pairing $\Phi(A, B) = \mathrm{tr}(AB)$ on $V = M_{2\times 2}(\mathbb{C})$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Suppose V and W are finite-dimensional inner product spaces and that $T : V \to W$ is linear.

   (a) Suppose $\mathbf{v} \in V$ is orthogonal to every vector in $\text{im}(T^*)$. Show that $\mathbf{v} \in \ker(T)$. [Hint: If $\mathbf{w} \in W$, write $\langle \mathbf{v}, T^*\mathbf{w} \rangle = \langle T\mathbf{v}, \mathbf{w} \rangle$.]

   (b) Show that the orthogonal complement of $\text{im}(T^*)$ in V is $\ker(T)$, and that the orthogonal complement of $\ker(T^*)$ in W is $\text{im}(T)$.

   - Remark: Observe that when T is a linear transformation on real vector spaces, part (d) reduces to our previous observation that the rowspace and nullspace of a matrix are orthogonal complements.

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5. Suppose $A \in M_{n\times n}(\mathbb{C})$. Prove that $\det(e^A) = e^{\text{tr}(A)}$, and deduce that e^A is always invertible. [Hint: Put A in Jordan canonical form and use the fact that matrix exponentiation respects conjugation.]

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6. Consider the sequence defined by the recurrence relation $a_{n+2} = 6a_{n+1} - 11a_n + 6a_{n-1}$ for $n \geq 1$, with $a_0 = a_1 = 2$ and $a_2 = 0$.

   (a) Show that $\begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 6 & -11 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \\ a_{n-1} \end{bmatrix}$, and conclude that $\begin{bmatrix} a_{n+2} \\ a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 6 & -11 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$.

   (b) Show that $A = \begin{bmatrix} 6 & -11 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is diagonalizable, and find an invertible matrix Q with $Q^{-1}AQ$ diagonal.

   (c) Use A^n to obtain a formula for a_n for any $n \geq 0$.

   (d) [Optional] Briefly describe a general procedure for solving an arbitrary linear recurrence of the form $x_n = c_{k-1}x_{n-1} + \cdots + c_0 x_{n-k}$ where the c_i are constants and k is a fixed positive integer.

   - Remark: Linear recurrences like the one described above are the discrete analogue of linear differential equations. This is one reason that we can use a very similar procedure to solve both differential equations and linear recurrences. One application is to give a formula for the famous Fibonacci numbers: if $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, then this method will eventually yield Binet's formula $F_n = \frac{1}{\sqrt{5}}(\varphi^n - \varphi^{-n})$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

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7. Suppose that Φ_1 and Φ_2 are two bilinear forms on V, and let $\beta \in F$.

   (a) Show that $\Phi_1 + \Phi_2$ is a bilinear form on V. Note that $\Phi_1 + \Phi_2$ is defined pointwise, so $(\Phi_1 + \Phi_2)(\mathbf{v}, \mathbf{w}) = \Phi_1(\mathbf{v}, \mathbf{w}) + \Phi_2(\mathbf{v}, \mathbf{w})$.

   (b) Show that $\beta\Phi_1$ is a bilinear form on V.

   (c) Show that $(\Phi_1 + \beta\Phi_2)^T = \Phi_1^T + \beta\Phi_2^T$. Deduce that if Φ_1 and Φ_2 are symmetric then so is $\Phi_1 + \beta\Phi_2$.

   (d) Show that the set of all bilinear forms on V is a vector space, and that the space of symmetric bilinear forms is a subspace of it.

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8. For $A, B \in M_{n\times n}(F)$, recall that we say A is congruent to B when there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that $B = Q^T AQ$. Prove that congruence is an equivalence relation on $M_{n\times n}(F)$.
