E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2020 ~ Homework 1, due Thu Jan 16th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

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**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false:
   (a) Every field has infinitely many elements.
   (b) The set of integers $\mathbb{Z}$ is not a field.
   (c) It is impossible to have $6 = 0$ in a field $F$.
   (d) Any system of linear equations has at least one solution.
   (e) There is a system of linear equations over $\mathbb{R}$ having exactly two different solutions.
   (f) For any $n \times n$ matrices $A$ and $B$, $(BA)^T = B^T A^T$.
   (g) For any invertible $n \times n$ matrices $A$ and $B$, $(A+B)^{-1} = A^{-1} + B^{-1}$.
   (h) For any invertible $n \times n$ matrices $A$ and $B$, $(BA)^{-1} = A^{-1}B^{-1}$.
   (i) If $A$ and $B$ are $n \times n$ matrices with $\det(A) = 2$ and $\det(B) = 3$, then $\det(AB) = 6$.
   (j) If $A$ is an $n \times n$ matrix with $\det(A) = 3$, then $\det(2A) = 6$.
   (k) For any $n \times n$ matrix $A$, $\det(A) = -\det(A^T)$.
   (l) For any $n \times n$ matrices $A$ and $B$, $\det(AB) = \det(B)\det(A)$.
   (m) If the coefficient matrix of a system of 6 linear equations in 6 unknowns is invertible, then the system has infinitely many solutions.
   (n) If $p$ and $q$ are polynomials in $F[x]$ of the same degree $n$, then $p + q$ also has degree $n$.
   (o) If $p$ and $q$ are polynomials in $F[x]$ of the same degree $n$, then $p \cdot q$ has degree $n^2$.

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2. Find the most general solution to each system of linear equations:

   (a) $\left\{ \begin{array}{r} x - 2y = 4 \\ 2x + 4y = 0 \end{array} \right\}$.

   (b) $\left\{ \begin{array}{r} x - 2y + 4z = 4 \\ 2x + 4y + 8z = 0 \end{array} \right\}$.

   (c) $\left\{ \begin{array}{r} x + 3y + z = -4 \\ -x - 6y + 8z = 10 \\ 2x + 4y + 8z = 0 \end{array} \right\}$.

   (d) $\left\{ \begin{array}{l} a + b + c + d \phantom{+e} = 2 \\ a + b + c \phantom{+d} + e = 3 \\ a + b \phantom{+c} + d + e = 4 \\ a \phantom{+b} + c + d + e = 5 \\ \phantom{a+} b + c + d + e = 6 \end{array} \right\}$.

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3. Compute the following things:
   (a) If $\mathbf{v} = (3, 0, -4)$ and $\mathbf{w} = (-1, 6, 2)$ in $\mathbb{R}^3$, find $\mathbf{v} + 2\mathbf{w}$, $||\mathbf{v}||$, $||\mathbf{w}||$, $||\mathbf{v} + 2\mathbf{w}||$, and $\mathbf{v} \cdot \mathbf{w}$.
   (b) The sum and product of the polynomials $2x + 3$ and $x^2 - 1$ in $\mathbb{R}[x]$.
   (c) The reduced row-echelon forms of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 & 2 & 3 \\ 2 & 1 & 0 & -1 & -2 \\ -4 & -2 & 0 & 3 & 0 \end{bmatrix}$.
   (d) The determinants of $\begin{bmatrix} -1 & 5 & 2 \\ 0 & -3 & 7 \\ 2 & 8 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 4 & 9 & 16 & 25 \\ 8 & 27 & 64 & 125 \end{bmatrix}$.

(e) The inverses of $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 1 & -3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -3 & -2 \\ -3 & 7 & 8 \\ 2 & -6 & -5 \end{bmatrix}$.

(f) The solution $X$ to $AX + B = C$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 0 \\ 2 & 1 \end{bmatrix}$.

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**Part II:** Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Let $\mathbf{v}$ and $\mathbf{w}$ be any vectors in $\mathbb{R}^n$.

   (a) Prove that $||\mathbf{v}+\mathbf{w}||^2 + ||\mathbf{v}-\mathbf{w}||^2 = 2\,||\mathbf{v}||^2 + 2\,||\mathbf{w}||^2$.

   (b) Prove that in any parallelogram, the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the four sides. [Hint: Suppose the sides are vectors $\mathbf{v}$ and $\mathbf{w}$.]

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5. Prove the following things:

   (a) Prove that $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{n\cdot(n+1)} = \dfrac{n}{n+1}$ for every positive integer $n$.

   (b) Prove that $1 + r + r^2 + r^3 + \cdots + r^n = \dfrac{1-r^{n+1}}{1-r}$ for any complex $r \neq 1$ and any positive integer $n$.

   (c) Prove that $(1+x)^n \geq 1 + nx$ for any real number $x > -1$ and any positive integer $n$.

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6. Suppose that $A$ and $B$ are $n \times n$ matrices with entries from a field $F$.

   (a) If $AB$ is invertible, show that $A$ and $B$ are invertible.

   (b) If $A$ is invertible and $AB$ is the zero matrix, show that $B$ must be the zero matrix.

   (c) If $A$ is invertible, show that $A^T$ is invertible and that its inverse is $(A^{-1})^T$.

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7. Let $F$ be a field of characteristic not 2 (i.e., in which $2 \neq 0$). A square matrix $A$ with entries from $F$ is called <u>symmetric</u> if $A = A^T$ and <u>skew-symmetric</u> if $A = -A^T$.

   (a) For any $n \times n$ matrix $B$, show that $B + B^T$ is symmetric and $B - B^T$ is skew-symmetric.

   (b) Show that any square matrix $M$ can be written uniquely in the form $M = S + T$ where $S$ is symmetric and $T$ is skew-symmetric.

   (c) If $A$ is a skew-symmetric $n \times n$ real matrix and $n$ is odd, show that $\det(A) = 0$.

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