

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each pair of integers (a, b) , use the Euclidean algorithm to calculate their greatest common divisor $d = \gcd(a, b)$ and also to find integers x and y such that $d = ax + by$.

(a) $a = 12, b = 44$. (b) $a = 5567, b = 12445$. (c) $a = 2019, b = 20223$. (d) $a = 377, b = 233$.

2. Find the prime factorizations of $a = 1001$, $b = 2019^{2020}$, $c = 2020^{2019}$, and $d = 12345654321$.
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3. It is sometimes claimed (occasionally in actual textbooks) that if p_1, p_2, \dots, p_k are the first k primes, then the number $n = p_1 p_2 \cdots p_k + 1$ used in Euclid's proof is always prime for any $k \geq 1$. Find a counterexample to this statement.
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Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

4. Show the following using only the axioms and basic properties of \mathbb{Z} :

(a) If a, b, c are integers, then $(b + c) + a = c + (a + b)$.
(b) If a, b are integers and $a \cdot b = a \cdot c$ then $a = 0$ or $b = c$.
(c) If a, b, c, x, y are integers where $a|b$ and $a|c$, then $a|(xb + yc)$.

5. The Fibonacci numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. The first few terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,

(a) Prove that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ for every positive integer n .
(b) Prove that $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$ for every positive integer n .
(c) Let $\varphi = \frac{1 + \sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1 - \sqrt{5}}{2}$. Prove that $F_n = \frac{1}{\sqrt{5}}(\varphi^n - \bar{\varphi}^n)$ for every positive integer n . [Hint: First show $\varphi^2 = \varphi + 1$ and likewise for $\bar{\varphi}$. Then write $F_{n+1} = F_n + F_{n-1}$ and use the formulas for the terms on the right-hand side.]

6. Prove that $\log_2 3$ is irrational. [Hint: Suppose otherwise, so that $\log_2 3 = a/b$. Convert this to statement about positive integers and find a contradiction.]
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7. Recall that the factorial of n is defined as $n! = n \cdot (n - 1) \cdots \cdot 1$, so for example $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. (Note that $0!$ is defined to be 1.)

(a) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ for every positive integer n .
(b) Prove that $n! + 1$ and $(n + 1)! + 1$ are relatively prime for every positive integer n . [Hint: Subtract a multiple of one from the other.]
(c) If $n \geq 2$, prove that the integers $n! + 2, n! + 3, \dots, n! + n$ are all composite. Deduce that there are arbitrarily large "prime gaps" (i.e., differences between consecutive prime numbers).
