Contents

0 Elliptic Curves and Modular Forms

These are lecture notes for the graduate course Math 7359: Elliptic Curves and Modular Forms, taught at Northeastern in Fall 2026.

0.1 (Sep 9) Overview + The Group Law

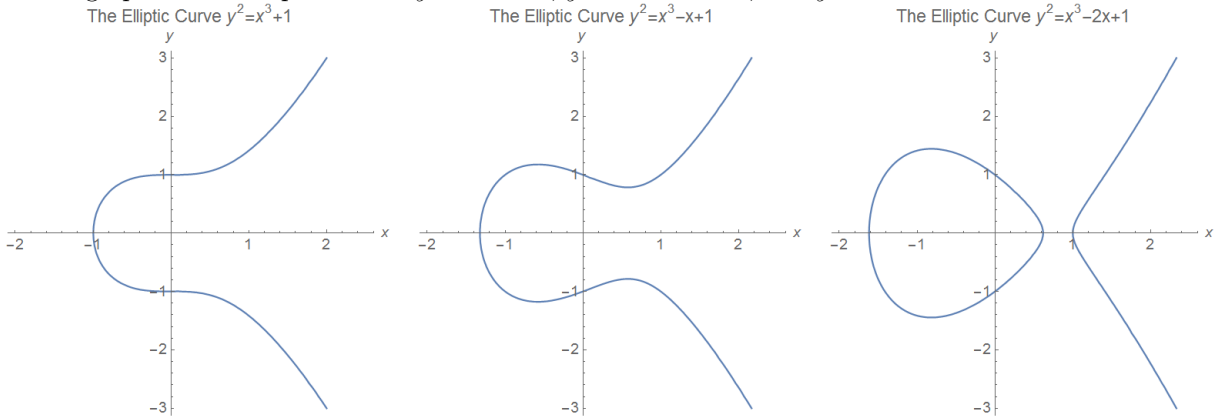
- The goal of this course is to give an overview of elliptic curves and modular forms, highlighting in particular the modern development of modularity, which establishes a very deep connection between these two otherwise very different classes of objects.
    - Elliptic curves are algebraic curves of genus 1 that arise in a wide variety of contexts in mathematics and their study involves techniques from nearly every discipline: algebra, analysis, geometry, topology, and (of course) number theory.
    - Modular forms are analytic functions on the complex upper half-plane satisfying a certain functional equation, and although they are intrinsically analytic objects, they turn out to have surprisingly deep connections to the (seemingly far more) algebraically-flavored elliptic curves.
    - In particular, the connection between elliptic curves and modular forms is central to Wiles's proof of Fermat's Last Theorem, and one of the goals at the end of the course is to elucidate some of the major ideas of this connection.
- In elementary coordinate geometry, one begins by studying the behavior of lines in the plane, which have the general equation $ax+by+c=0$, and then afterwards studies quadratic curves (i.e., the conic sections) having the general equation $ax^2+bxy+cy^2+dx+ey+f=0$.
    - In each case, we often perform simple algebraic manipulations and changes of variable to put the equations into a more standard form.
    - For example, if $b \neq 0$, we can rewrite $ax+by+c=0$ as $y=(-a/b)x+(-c/b)$, which for $m=-a/b$ and $b'=-c/b$ has the more familiar form $y=mx+b'$.
    - Similarly, if $a \neq 0$, we can perform a change of variable $x'=y+(b/(2a))x$ in the equation $ax^2+bxy+cy^2+dx+ey+f=0$ to remove the cross term bxy, and then we can complete the square in x and in y and then rescale the variables to obtain an equation of the form $x^2 \pm y^2 = 1$ or $y=x^2$, depending on which quadratic coefficients are zero.
- Our goal now is to study cubic curves in the plane, which have the general form $ax^3+bx^2y+cxy^2+dy^3+ex^2+fxy+gy^2+hx+iy+j=0$.
    - Like in the case of quadratic curves above, we can perform various changes of variable to reduce the general form to a simpler one.
    - We will not give the full details of the procedure now, as it relies on some facts about cubic curves that we will prove later.

- Instead, we will summarize matters by saying that as long as the equation is actually cubic (i.e., it is not the case that all of a, b, c, d are zero), then the general equation above can always be transformed using rational changes of variable into one of the form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, for appropriate coefficients a_1, a_2, a_3, a_4, a_6.

- Definition: An elliptic curve E over a field K is a curve having an equation of the form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, for appropriate coefficients a_1, a_2, a_3, a_4, a_6 in K. This expression is called the Weierstrass form of E.

  - This expression is not the simplest possible one: as long as the characteristic of K is not 2 or 3, we can simplify it further by completing the square in y and completing the cube in x.
  - Explicitly, if we set $y' = y + (a_1/2)x + (a_3/2)$ and $x' = x + (a_2/3)$, we can reduce the Weierstrass equation above to one of the form $(y')^2 = (x')^3 + A(x') + B$.
  - An elliptic curve having an equation of the form $y^2 = x^3 + Ax + B$ is sometimes said to be in "reduced" Weierstrass form.
  - This reduced form is much more amenable for computations, and (in fact) it is nearly unique: the only change of variables that preserves it is one of the form $x = u^2x'$, $y = u^3y'$ for some nonzero u, from which we see that $A = u^4A'$ and $B = u^6B'$.

- Here are graphs of the elliptic curves $y^2 = x^3 + 1$, $y^2 = x^3 - x + 1$, and $y^2 = x^3 - 2x + 1$ over $\mathbb{R}$:

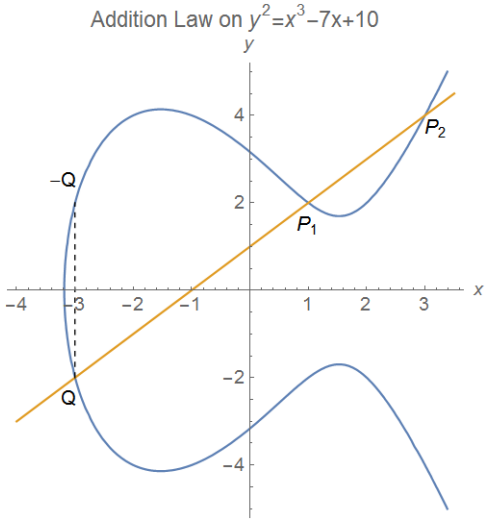


  - In general, we can see that the graph of an elliptic curve $y^2 = x^3 + Ax + B$ will always be symmetric about the x-axis, since if (x, y) satisfies the equation then so does $(x, -y)$. The graph will also have either one or two connected components according to the number of real roots that $x^3 + Ax + B$ has.
  - Exercise: Show that graph of an elliptic curve over $\mathbb{R}$ will have two components when the polynomial $x^3 + Ax + B$ has three distinct real roots, and will have one component otherwise.
  - Notice also that the tangent line at each crossing of the x-axis is vertical for each curve above. Using implicit differentiation, we can compute $y' = \dfrac{3x^2 + A}{2y}$: thus, we see that $y' = \infty$ when y is zero, provided that $3x^2 + A$ is not also zero. This behavior can only occur when $x^3 + Ax + B$ has a root in common with its derivative $3x^2 + A$, which is in turn equivalent to saying that $x^3 + Ax + B$ has a double root.

- Definition: If the polynomial $x^3 + Ax + B$ has a repeated root, we say that the elliptic curve $y^2 = x^3 + Ax + B$ is singular. Otherwise (if the roots are distinct) we say the elliptic curve is nonsingular. A curve is singular if and only if its discriminant $\Delta = -16(4A^3 + 27B^2)$ is zero.

  - The second statement follows from the observations above: the polynomial $x^3 + Ax + B$ has a repeated root if and only if it has a root in common with its derivative $3x^2 + A$. This occurs precisely when $x^2 = -A/3$, from which we see that $x(2A/3) + B = 0$ so $x = -3B/(2A)$: then substituting for x yields $\Delta = 0$ almost immediately.
  - Remark: The presence of the constant -16 is superfluous here, but there is also a definition of Δ in terms of the original coefficients a_1, a_2, a_3, a_4, a_6 for a general Weierstrass form. To avoid having denominators in that expression, we end up needing an extra factor of -16 in the one we gave above.

- The core property of elliptic curves that makes them so interesting is that if we have two points that lie on the curve, we can use them to construct a third point on the curve.
  - Explicitly, suppose $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are two distinct points on an elliptic curve E: $y^2 = x^3 + Ax + B$.
  - Draw the line through P_1 and P_2: we claim that this line L must intersect E in a third point Q.
  - To see this, suppose the line through P_1 and P_2 has equation $y = mx + b$. (We are tacitly excluding the possibility that the line is vertical, but we will come back to this case in a moment.)
  - Then the intersection points between L and E are the solutions to the system $y = mx + b$ and $y^2 = x^3 + Ax + B$. Equivalently, we must solve $(mx+b)^2 = x^3 + Ax + B$, or $x^3 + (-m^2)x^2 + (A - 2mb)x + (B - b^2) = 0$.
  - However, we already know that this cubic has the two roots $x = x_1$ and $x = x_2$, so it must have a third root: this gives us the third point Q we wanted.
- Once we construct a third point on an elliptic curve this way, we might try to find more points.
  - If we try this procedure directly using our points P_1, P_2, and Q, however, we will not get anywhere: the line through any of these two points intersects the elliptic curve at the other point.
  - However, we can also exploit the vertical symmetry of the curve to make new points: if $P = (x, y)$ lies on the curve, then the point $-P = (x, -y)$ also lies on the curve.
  - If we combine these two procedures, we can often generate many points on the curve starting from just two.
- Definition (Group Law I): If P_1 and P_2 are two distinct points on the elliptic curve $E : y^2 = x^3 + Ax + B$, let $Q = (x', y')$ be the third intersection point of E with the line L joining P_1 and P_2. We define the sum $P_1 + P_2$ to be the point $-Q = (x', -y')$.
  - It is not immediately clear why we define the sum of two points to be the reflection of Q rather than Q itself. This will become clearer in a moment.
  - Note that if we attempt to add two points which are vertical reflections of one another on the graph of $y^2 = x^3 + Ax + B$, the resulting line will not intersect the curve again.
  - To remedy this, we declare that the curve also includes a point at ∞, which we denote simply as ∞, that we consider as lying on any vertical line. (What we are really doing here is working with the projective model of the curve, rather than the affine one.)
- Example: Given the points $P_1 = (1, 2)$ and $P_2 = (3, 4)$ on the elliptic curve $y^2 = x^3 - 7x + 10$, find the sums $P_1 + P_2$ and $(P_1 + P_2) + P_2$.
  - It is easy to verify that both points lie on the curve. Here is a plot of the curve and the line $y = x + 1$ through the two points:

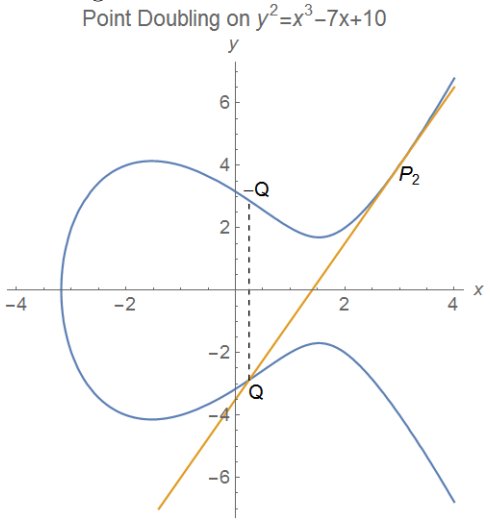


- The point Q lies on the intersection of $y = x + 1$ and $y^2 = x^3 - 7x + 10$, so $(x+1)^2 = x^3 - 7x + 10$.
- This equation is equivalent to $x^3 - x^2 - 9x + 9 = 0$, which factors as $(x-1)(x-3)(x+3) = 0$. Then the x-coordinate of Q is -3 so $Q = (-3, -2)$.
- Thus, the sum $P_1 + P_2$ is the vertical reflection of Q, which is $\boxed{(-3, 2)}$.
- To find the sum $(P_1 + P_2) + P_2$ we perform a similar procedure: the line through $P_1 + P_2$ and P_2 has equation $y = \frac{1}{3}x + 3$.
- Then we must solve $(\frac{1}{3}x + 3)^2 = x^3 - 7x + 10$, or $x^3 - \frac{1}{9}x^2 - 9x + 1 = 0$.
- Factoring yields $(x - \frac{1}{9})(x+3)(x-3) = 0$, so $Q' = (\frac{1}{9}, \frac{82}{27})$, and thus $(P_1 + P_2) + P_2 = \boxed{(\frac{1}{9}, -\frac{82}{27})}$.

- We would also like to be able to add a point to itself.
  - It is straightforward to see from our definition that if P_1 and P_2 are distinct points, then $P_1 + P_2$ is a continuous function of the coordinates of the points.
  - If we are working over $\mathbb{R}$, or another field where limits exist, we could define the addition $P + P$ to be the limit as $P_1 \to P$ of sums $P + P_1$. Geometrically, the lines used in the construction also have a limit as $P \to P_1$: they approach the tangent line to the curve E at the point P.
  - Since we want a definition over any field, we define $P + P$ by letting L be the tangent line to E at P, and then taking Q to be the third point of intersection of L with E.

- Definition (Group Law II): If P is any point on the elliptic curve $E : y^2 = x^3 + Ax + B$, let $Q = (x', y')$ be the third intersection point of E with the tangent line L to E at P. We define the sum $P + P$ to be the point $-Q = (x', -y')$.

- Example: Given the points $P_1 = (1, 2)$ and $P_2 = (3, 4)$ on the elliptic curve $y^2 = x^3 - 7x + 10$, find the sums $P_2 + P_2$ and $(P_1 + P_2) + P_2$.
  - Differentiating implicitly yields $2yy' = 3x^2 - 7$ so that $y' = (3x^2 - 7)/(2y)$. Thus, the tangent line to E at P_2 has slope $\frac{5}{2}$ and its equation is $y = \frac{5}{2}x - \frac{7}{2}$.
  - Here is a plot of the curve and the tangent line at P_2:



  - The point Q lies on the intersection of $y = \frac{5}{2}x - \frac{7}{2}$ and $y^2 = x^3 - 7x + 10$, so $(\frac{5}{2}x - \frac{7}{2})^2 = x^3 - 7x + 10$.
  - This equation is equivalent to $x^3 - \frac{25}{4}x^2 + \frac{21}{2}x - \frac{9}{4} = 0$, which factors as $(x - \frac{1}{4})(x-3)(x-3) = 0$.
  - Then the x-coordinate of Q is $1/4$ so $Q = (\frac{1}{4}, -\frac{23}{8})$, and so $P_2 + P_2 = \boxed{(\frac{1}{4}, \frac{23}{8})}$.

- To find the sum $P_1 + (P_2 + P_2)$ we then find the sum of $P_1 = (1, 2)$ with $(\frac{1}{4}, \frac{23}{8})$. The line through these points is $y = -\frac{7}{6}x + \frac{19}{6}$.
- Then we must solve $(-\frac{7}{6}x + \frac{19}{6})^2 = x^3 - 7x + 10$, which has solutions $x = \frac{1}{9}, \frac{1}{4}, 1$.
- Then $Q' = (\frac{1}{9}, \frac{82}{27})$, and thus $P_1 + (P_2 + P_2) = \boxed{(\frac{1}{9}, -\frac{82}{27})}$.

- Note that in the previous two examples, we computed $(P_1 + P_2) + P_2 = (\frac{1}{9}, -\frac{82}{27}) = P_1 + (P_2 + P_2)$, and so we see in this case that the addition law is actually associative. Much more is true:
- Theorem (Group Law): If K is any field and E is any elliptic curve defined over K, then under the addition law defined above, the set of K-valued points on E forms an abelian group with identity ∞ and with the inverse of any point P given by $-P$.
  - We will give arguments for an elliptic curve of the form $y^2 = x^3 + Ax + B$, but the theorem holds in full generality for any elliptic curve.
  - Proof: The addition law is commutative, since the line used in computing $P_1 + P_2$ and $P_2 + P_1$ is the same in each case.
  - To see that ∞ is an identity, consider the sum $P + \infty$. The line passing through P and ∞ is the vertical line through P which also intersects E at the point $-P$. Then by the geometric definition, $P + \infty = -(-P) = P$.
  - To see that $-P$ is an inverse of P, observe that the line passing through P and $-P$ is a vertical line, so the other point on it is ∞. The reflection of ∞ is also ∞, so $P + (-P) = \infty$.
  - Associativity of the addition law is the only nontrivial result in this theorem. It can be done with a tedious algebraic computation using explicit formulas for the addition law (see below).
  - We give another argument using the following consequence of basic linear algebra: if C_1 and C_2 are two distinct plane cubics intersecting in 9 points, then any other cubic D passing through 8 of them must be a linear combination of C_1 and C_2 hence also pass through the 9th point.
  - To prove this we merely observe that the vector space of all equations of cubic curves is 10-dimensional and each point imposes one linear condition, so the space of equations passing through 8 given points is 2-dimensional by the nullity-rank theorem (this also requires checking that the linear conditions are independent, which we omit). Since C_1 and C_2 are distinct their equations yield a basis for this space, and so any other cubic D is a linear combination. In particular, then, since the 9th point satisfies the equation for both C_1 and C_2, it also satisfies the equation for any linear combination.
  - So suppose P_1, P_2, P_3 are points on an elliptic curve E. Construct the following lines:
    L_1 through P_1, P_2, S $\quad$ L_2 through $-S$, P_3, T $\quad$ L_3 through ∞, U, $-U$
    M_1 through $\infty, S, -S$ $\quad$ M_2 through P_2, P_3, U $\quad$ M_3 through $-U$, P_1, T'
  - Then $-T = (P_1 + P_2) + P_3$ and $-T' = P_1 + (P_2 + P_3)$, so we wish to show that $T = T'$.
  - Let C_1 be the cubic $L_1L_2L_3$ and C_2 be the cubic $M_1M_2M_3$ (by which we mean, write down the cubic equation $(ax+by+c)(a'x+b'y+c')(a''x+b''y+c'') = 0$ obtained by multiplying together the corresponding equations for the lines).
  - Then C_1 and E both pass through the 9 points P_1, P_2, P_3, S, $-S$, ∞, U, $-U$, and T. Since C_2 also passes through the first 8 of these points, it must also pass through the 9th, which is T.
  - But since C_2 and E can only intersect in at most 9 points by Bézout's theorem[1], and these 9 points are P_1, P_2, P_3, S, $-S$, ∞, U, $-U$, and T', we must have $T' = T$.

- For convenience in doing numerical computations, we will also write down the general formula for the group law on any reduced Weierstrass curve:

[1] Bézout's theorem states that two plane curves of degrees m and n not sharing a common component will intersect in mn points over an algebraically closed field, counting multiplicities. Applied when $m = n = 3$, we see that two plane cubics intersect in 9 points (over an algebraically closed field, counting multiplicities).

- Proposition (Explicit Group Law): Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be points on the elliptic curve $E : y^2 = x^3 + Ax + B$. Then $P_1 + P_2 = (x_3, y_3)$ where $x_3 = m^2 - x_1 - x_2$ and $y_3 = -m(x_3 - x_1) - y_1$, with $m = \begin{cases} (y_2 - y_1)/(x_2 - x_1) & \text{if } P_1 \neq P_2 \\ (3x_1^2 + A)/(2y_1) & \text{if } P_1 = P_2 \end{cases}$. If m is infinite, then $P_1 + P_2 = \infty$.
  - Observe in particular that the addition formula is rational, in the sense that the result is always a rational function of the inputs. In particular, the sum of two points whose coordinates lie in a field K will also lie in K.
  - Proof: If $P_1 \neq P_2$ then the line joining P_1 and P_2 has equation $y - y_1 = m(x - x_1)$ where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.
  - We therefore obtain the equation $(mx - mx_1 + y_1)^2 = x^3 + Ax + B$, which has the form $x^3 - m^2x^2 + Cx + D = 0$ for appropriate constants C and D.
  - The polynomial $x^3 - m^2x^2 + Cx + D$ must factor as $(x - x_1)(x - x_2)(x - x_3)$, so upon multiplying out we see that $x_1 + x_2 + x_3 = m^2$. This yields the stated value of x_3, and then $y_3 = m(x_3 - x_1) + y_1$ (where we have multiplied by -1 to account for the vertical reflection).
  - If $P_1 = P_2$ then everything is the same, except instead m is the slope of the tangent line at P_1. By implicit differentiation, we see that $2yy' = 3x^2 + A$ so $m = \dfrac{3x_1^2 + A}{2y_1}$ here, as claimed.
- Example: If $P_1 = (1, 3)$ and $P_2 = (0, 2)$ on the elliptic curve $y^2 = x^3 + 4x + 4$ over $\mathbb{F}_5$ (i.e., with coefficients mod 5), find $P_1 + P_2$ and $P_1 + P_1$.
  - We simply apply the appropriate formulas: adding $Q_1 = (x_1, y_1)$ to $Q_2 = (x_2, y_2)$ produces (x_3, y_3) where $x_3 = m^2 - x_1 - x_2$ and $y_3 = -m(x_3 - x_1) - y_1$, and $m = \begin{cases} (y_2 - y_1)/(x_2 - x_1) & \text{if } Q_1 \neq Q_2 \\ (3x_1^2 + A)/(2y_1) & \text{if } Q_1 = Q_2 \end{cases}$.
  - With $(x_1, y_1) = (1, 3)$ and $(x_2, y_2) = (0, 2)$ we obtain $m = \dfrac{2 - 3}{0 - 1} = 1$, so $x_3 = 0$ and $y_3 = -1(0-1)-3 = 3$, so $P_1 + P_2 = \boxed{(0, 3)}$.
  - Likewise, with $(x_1, y_1) = (x_2, y_2) = (1, 3)$ we obtain $m = \dfrac{3 + 4}{2 \cdot 3} = 2$, so $x_3 = 2$ and $y_3 = -2(2-1)-3 = 0$, so $P_1 + P_1 = \boxed{(2, 0)}$.
- We will also remark that there are formulas for the addition law on a more general elliptic curve $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
  - We use the same geometric construction as for an elliptic curve in reduced Weierstrass form $y^2 = x^3 + Ax + B$, namely, by taking the additive inverse of a point P to be the other point on the (vertical) line joining P to ∞, and by taking the sum $P_1 + P_2$ to be the additive inverse of the other point on the line joining P_1 and P_2 (which is the tangent line when $P_1 = P_2$).
- (Tedious) Exercise: Suppose E is an elliptic curve with a Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ with $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on E. Show that the additive inverse is given by $-P_1 = (x_0, -y_0 - a_1x_0 - a_3)$, and the sum $P_1 + P_2$ is given by ∞ when $x_1 = x_2$ and $y_1 = -y_2 - a_1x_2 - a_3$ and by (x_3, y_3) where $x_3 = m^2 + a_1m - a_2 - x_1 - x_2$ and $y_3 = -(m + a_1)x_3 - b - a_3$ where $y = mx + b$ is the line joining P_1 and P_2 (or the tangent line when $P_1 = P_2$), which explicitly has $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, $b = \dfrac{x_2y_1 - x_1y_2}{x_2 - x_1}$ when $P_1 \neq P_2$ and has $m = \dfrac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3}$ and $b = \dfrac{-x_1^3 + a_4x_1 + 2a_6 - a_3y_1}{2y_1 + a_1x_1 + a_3}$ when $P_1 = P_2$.

0.2 (Sep 14) The Group Structure of Elliptic Curves, Nagell-Lutz

- Now that we have established the group law, we can now (attempt to) compute the abelian group structure of the set of points on a given elliptic curve over a field K.
- Definition: If E is an elliptic curve with coefficients lying in a field K, we define the set $E(K)$, the K-rational points on E, to be the set of points on E whose entries lie in K, along with the point ∞.

- When K is finite, clearly $E(K)$ must also be finite, in which case (in principle) we can simply list all of the elements of $E(K)$ and write down the group structure explicitly.

- Example: Find all of the points on the elliptic curve $y^2 = x^3 + 4x + 4$ over $\mathbb{F}_3$ and identify the group structure explicitly.
  - By simply computing $x^3 + 4x + 4$ for each $x \in \mathbb{F}_3$ and testing which are squares we can see that there are 4 points on E: $(0,1)$, $(0,2)$, $(1,0)$, and ∞.
  - The group of points is therefore either cyclic and isomorphic to $\mathbb{Z}/4\mathbb{Z}$, or isomorphic to the Klein 4-group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, but since $(0,1) + (0,1) = (1,0)$ is not the identity, in fact the group must be cyclic and generated by $(0,1)$.

- Exercise: Pick an elliptic curve in Weierstrass form (e.g., $y^2 = x^3 + 4x + 1$) and after checking whether it is nonsingular, find all of its points over $\mathbb{F}_3$, $\mathbb{F}_5$, $\mathbb{F}_7$, $\mathbb{F}_{11}$, and $\mathbb{F}_{13}$, and identify the group structure explicitly in each case.

- We have some additional notation useful when computing orders of elements:

- Definition: If P is a point on an elliptic curve E, we define the multiples of P as $[n]P = \underbrace{P + P + \cdots + P}_{n \text{ terms}}$ for positive integers n, with $[0]P = \infty$ and $[-n]P = -[n]P$. The subgroup of $E(K)$ generated by P is then simply the set $\{[n]P : n \in \mathbb{Z}\}$, and as usual the order of P is the cardinality of this set.
  - Trivially, ∞ is the only point of order 1 on E.
  - The first nontrivial case is to identify the points of order 2: these points satisfy $P+P = \infty$. Geometrically, this means that the tangent line to the graph of E at P passes through $-\infty = \infty$, meaning that the tangent line at P is vertical. From the explicit formula $2yy' = 3x^2 + A$ we see that this is, in turn, equivalent to saying that $y = 0$.
  - Therefore, the points (x, y) of order 2 are those having $y = 0$. Since this requires $x^3 + Ax + B = 0$, we see that there are at most 3 such points.
  - For points of order 3, we see that such points P satisfy $P + P + P = \infty$ so that $P + P = -P$, which means that the third intersection point of the tangent line to E at P also goes through P. Equivalently, this says that the point P is an inflection point of the curve.

- We can be more precise if we work with the subgroup of $E(K)$ consisting of all m-torsion points together:

- Definition: The m-torsion subgroup of an elliptic curve E defined over K is the kernel of the multiplication-by-m map (i.e., the points $P \in E(K)$ with $[m]P = \infty$) and is denoted $E_K[m]$.
  - Later, we will also be interested in the full group of m-torsion points on E when we consider E as being defined over the algebraic closure $\overline{K}$: this full m-torsion group is denoted $E[m]$.

- Example: Find the points of order 2 on the elliptic curve $E : y^2 = x^3 + x$ over $\mathbb{Q}$ and over $\mathbb{C}$, and identify the group structure of the 2-torsion group $E[2]$ over each field.
  - From the discussion above, the 2-torsion points are the points with $y = 0$, which requires $x^3 + x = 0$ so that $x = 0$ or $x = \pm i$.
  - Over $\mathbb{Q}$, there is therefore one 2-torsion point $\boxed{(0,0)}$. Then the 2-torsion group $E_{\mathbb{Q}}[2]$ is $\{\infty, (0,0)\}$ and its group structure is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
  - Over $\mathbb{C}$ we have three 2-torsion points: $\boxed{(0,0),\ (i,0),\ (-i,0)}$. Then the 2-torsion group $E_{\mathbb{Q}}[2]$ is $\{\infty, (0,0), (i,0), (-i,0)\}$ Since all of the nontrivial elements in this group have order 2, the group structure is isomorphic to the Klein 4-group $V_4 \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

- For points of higher order, it is more difficult to give nice geometric or algebraic descriptions of $E[m]$ directly from the definition.

- As we will show later using complex lattices, for any elliptic curve defined over (a subfield of) $\mathbb{C}$, the m-torsion subgroup $E[m]$ is always isomorphic to $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$, and the same is true more generally in any field of characteristic zero.

- The following theorem of Nagell and Lutz provides a convenient way to calculate the torsion points on any elliptic curve over $\mathbb{Q}$:

- Theorem (Nagell-Lutz): Suppose E is an elliptic curve over $\mathbb{Q}$ with Weierstrass equation $y^2 = x^3 + Ax + B$ where A and B are integers, and let $D = -4A^3 - 27B^2$ be the reduced discriminant of E. If $P = (x, y)$ is a rational point of finite order, then x and y are integers. Furthermore, either $y = 0$ or y^2 divides D.

  - Exercise: Show that by making an appropriate change of variables, any rational Weierstrass form can be converted into one with A, B integers. Illustrate by finding a Weierstrass form with integer coefficients for $y^2 = x^3 + \frac{3}{2}x + \frac{2}{5}$.
  - We emphasize here that the Nagell-Lutz theorem is not an if-and-only-if: there can exist points (x, y) with y dividing D that do not have finite order.
  - To prove the theorem we will assemble a few preliminary lemmas.

- Lemma 1: Let $P = (x, y)$ be a point on an elliptic curve E over $\mathbb{Q}$ such that P and $[2]P$ both have integral coordinates. Then either $y = 0$ or y^2 divides the reduced discriminant of E.

  - Proof: If $[2]P = \infty$ then from our discussion of 2-torsion points we see that $y = 0$.
  - Otherwise assume $[2]P \neq \infty$. If E has Weierstrass equation $y^2 = x^3 + Ax + B$, then by the explicit group law formula the x-coordinate of $[2]P$ is $\dfrac{(3x^2 + A)^2}{4y^2} - 2x$. Since this quantity is an integer by hypothesis, we must have $y | (3x^2 + A)$.
  - Now we invoke the identity $D = 27(x^3 + Ax - B)(x^3 + Ax + B) - (3x^2 + 4A)(3x^2 + A)^2$, which can be derived by applying the Euclidean algorithm in $\mathbb{Z}[x]$ to $x^3 + Ax + B$ and the square of its derivative $(3x^2 + A)^2$.
  - Since y^2 divides both $x^3 + Ax + B = y^2$ and $(3x^2 + A)^2$, it divides Δ, as claimed.

- We now record some facts about the coordinates of $[m]P$:

- (Tedious) Exercise: Let E be an elliptic curve and $P = (x, y)$ be a point on E. Define the polynomials $\varphi_0 = 0$, $\varphi_1 = 1$, $\varphi_2 = 2y$, $\varphi_3 = 3x^4 + 6Ax^2 + 12Bx - A^2$, $\varphi_4 = 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3)$, and in general $\varphi_{2n+1} = \varphi_{n+2} \cdot \varphi_n^3 - \varphi_{n-1}\varphi_{n+1}$ and $\varphi_{2n} = \dfrac{\varphi_n}{2y} \cdot (\varphi_{n+2}\varphi_{n-1}^3 - \varphi_{n-2}\varphi_{n+1}^2)$ for $n \geq 2$.

  1. With $y^2 = x^3 + Ax + B$, show that φ_n can be written as a polynomial in $\mathbb{Z}[x, A, B]$ when n is odd and can be written as y times a polynomial in $\mathbb{Z}[x, A, B]$ when n is even.
  2. Show that φ_n^2 is a polynomial of degree $n^2 - 1$ in x with leading coefficient n^2 while $x\varphi_n^2 - \varphi_{n-1}\varphi_{n+1}$ is a polynomial of degree n^2 in x with leading coefficient 1.
  3. Show that the coordinates of $[n]P$ are (x_n, y_n) where $x_n = \dfrac{x\varphi_n^2 - \varphi_{n-1}\varphi_{n+1}}{\varphi_n^2}$ and $y_n = \dfrac{\varphi_{n+2}\varphi_{n-1}^2 - \varphi_{n-2}\varphi_{n+1}^2}{4y\varphi_n^3}$.

- Lemma 2: If E is an elliptic curve over $\mathbb{Q}$ and P is a point on E such that $[m]P$ has integral coordinates for some $m \geq 1$, then P itself has integral coordinates.

  - Proof: Suppose that $P = (s/t, y)$ where s and t are relatively prime and $t > 0$. By the exercise above, x_m is the quotient of a monic polynomial of degree n^2 in x by a polynomial of degree at most $n^2 - 1$ in x.
  - This means that the value of the numerator polynomial $x\varphi_n^2 - \varphi_{n-1}\varphi_{n+1}$ is of the form a/t^{n^2} with a, t relatively prime, while the denominator polynomial is of the form b/t^{n^2-1} for some b. But then x_m is of the form $a/(tb)$ which cannot be an integer unless $t = 1$. Then the x-coordinate of P is integral, so then since $y^2 = x^3 + Ax + B$ is an integer and y is rational, y is also integral, as required.

- Lemma 3: If $P = (x, y)$ has rational coordinates on $E : y^2 = x^3 + Ax + B$, then $(x, y) = (q/d^2, s/d^3)$ for some positive integer d and some integers p, r relatively prime to d.
  - Proof: Letting $x = q/r$ and $y = s/t$ in lowest terms with $r, t > 0$ and clearing denominators in $y^2 = x^3 + Ax + B$ yields $s^2r^3 = t^2(q^3 + Ar^2p + Br^3)$.
  - Then r is relatively prime to $q^3 + Ar^2p + Br^3$ hence r^3 divides t^2, and likewise t is relatively prime to s^2 hence t^2 divides r^3. Thus $t^2 = r^3$ so letting $d = t/r$ we see $r = d^2$ and $t = d^3$, as required.
- We can now assemble the results for a proof of Nagell-Lutz:
  - Proof (of Nagell-Lutz): Suppose $P = (x, y)$ is a torsion point on E. We first show that P has integer coordinates.
  - Let p be a prime divisor of m and consider $Q = [m/p]P$, which is a torsion point of order p. By Lemma 2 it suffices to show that $Q = (x_Q, y_Q)$ has integer coordinates, since this would imply that P itself has integer coordinates.
  - If $p = 2$ then $[2]Q = \infty$ which requires $y_Q = 0$ so that $x_Q^3 + Ax_Q + B = 0$ so that x_Q is integral by the rational root test; then y_Q is also necessarily integral since it is rational and its square is the integer $x_Q^3 + Ax_Q + B$.
  - Otherwise suppose p is odd. Since $[p]Q = \infty$, the denominator term φ_p of the x-coordinate must vanish when evaluated at x_Q. But for fixed integers A and B and odd p, φ_p is a polynomial in x with integer coefficients of degree $(p^2 - 1)/2$ and leading coefficient p.
  - By Lemma 3, in lowest terms we have $x_Q = q/d^2$ for some integer d, so since $\varphi_p(x) = px^{(p^2-1)/2} + O(x^{(p^2-3)/2})$, we see that $d^{p^2-1}\varphi_p(q/d^2) = p \cdot q^{(p^2-1)} + d^2 \cdot k$ for an integer k. Since this quantity must be zero and p is prime, this requires d^2 to divide p, and hence $d = 1$. This means x_Q is an integer, hence so is y_Q by the argument used above. Thus Q is integral and hence P is integral by Lemma 2.
  - By repeating the same argument for $[2]P$ we see that $[2]P$ is also integral. Then, finally, by Lemma 1, we conclude that either $y = 0$ or y^2 divides the reduced discriminant of E, as desired.