E. Dummit's Math 4527 ~ Number Theory 2, Fall 2026 ~ Homework 2, due Fri Sep 25th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. **Use of generative AI in any manner is not allowed on this or any other course assignments.**

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $\alpha = [\overline{3}] = [3, 3, 3, 3, 3, 3, 3, \dots]$.
   (a) Find α.
   (b) The first convergent to α is 3. Find the next five convergents to α.
   (c) For the sequence defined by $s_0 = 1$, $s_1 = 3$, and for $n \geq 2$, $s_{n+1} = 3s_n + s_{n-1}$, find $\lim_{n\to\infty} s_{n+1}/s_n$.

---

2. Express the following continued fractions as real numbers:
   (a) $[3, 1, 4, 1, 5]$.
   (b) $[\overline{1, 2, 3}]$.
   (c) $[\overline{3, 2, 1}]$.
   (d) $[\overline{3, 1, 2}]$.
   (e) $[3, \overline{1, 2}]$.
   (f) $[1, 2, \overline{1, 9, 1}]$.

---

3. Find the continued fraction expansion, and the first five convergents, for each of the following:
   (a) $\sqrt{3}$.
   (b) $\sqrt{11}$.
   (c) $\dfrac{4 + \sqrt{13}}{5}$.

---

4. Find the rational number with denominator less than N closest to each of the following real numbers α:
   (a) $\alpha = \sqrt{13}$, $N = 100$.
   (b) $\alpha = \sqrt{2}$, $N = 100$.
   (c) $\alpha = e$, $N = 10000$. [Hint: See problem 8 for the continued fraction expansion of e.]

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. The goal of this problem is to give a method for manipulating continued fractions using linear algebra. So suppose $a_0, a_1, \dots, a_n, \dots$ is a sequence of positive integers and set $p_n/q_n = [a_0, a_1, \dots, a_n]$ for each n.
   (a) Prove that $\begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$.
   (b) Show that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$. [Hint: Determinant.]
   (c) Show that $[a_n, a_{n-1}, \dots, a_0] = p_n/p_{n-1}$ and that $[a_n, a_{n-1}, \dots, a_1] = q_n/q_{n-1}$. [Hint: Transpose.]

---

6. Let α be the real number $\alpha = \sum_{n=1}^{\infty} \frac{n^2}{100^n} = 0.0104091625364964....$

   (a) Find a rational number p/q with denominator $< 10^6$ that agrees with the expansion of α above, to the 16 decimal places shown.

   (b) Prove that your answer in part (a) is the only rational number with denominator less than 10^{10} whose decimal expansion agrees with that of α to the 16 decimal places shown. [Hint: If $|p/q - a/b| < 10^{-16}$ then $|pb - aq| < 10^{-16}bq$.]

---

7. The goal of this problem is to establish some estimates on the nth convergent denominator q_n, where $p_n/q_n = [a_0, a_1, \dots, a_n]$.

   (a) Show that $q_n \geq \varphi^{n-1}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

   (b) Suppose that all terms $a_i \leq M$. Show that $q_n \leq \alpha^n$ for $\alpha = \frac{M+\sqrt{M^2+4}}{2}$.

---

8. The goal of this problem is to give another proof of Dirichlet's Diophantine approximation theorem (indeed, this is essentially Dirichlet's original proof, and it represents the first recognized use of the pigeonhole principle). Let α be an irrational real number.

   (a) Let n be a positive integer and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Show that some two of 0, $\{\alpha\}$, $\{2\alpha\}$, ... , $\{n\alpha\}$, 1 must be within $\frac{1}{n+1}$ of each other. [Hint: Pigeonhole.]

   (b) Let n be a positive integer. Show that there exists a positive integer $q \leq n$ and an integer p such that $|q\alpha - p| < \frac{1}{n+1}$. [Hint: If $c\alpha$ and $d\alpha$ have fractional parts near each other, consider $(c - d)\alpha$.]

   (c) Show that there exist infinitely many pairs of integers (p, q) such that $|q\alpha - p| < 1/q$.

---

9. [Challenge] The goal of this problem is to obtain the continued fraction expansion $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \dots]$. Let $\beta = [1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, \dots]$ be the real number with continued fraction terms $a_{3i} = 1$, $a_{3i+1} = 2i$, and $a_{3i+2} = 1$ for each $i \geq 0$, and let $C_i = p_i/q_i$ be its convergents.

   (a) Show that the convergents $C_i = p_i/q_i$ have numerators and denominators satisfying the recurrences

$$\begin{aligned} p_{3n} &= p_{3n-1} + p_{3n-2} & \qquad q_{3n} &= q_{3n-1} + q_{3n-2} \\ p_{3n+1} &= 2np_{3n} + p_{3n-1} & \qquad q_{3n+1} &= 2nq_{3n} + q_{3n-1} \\ p_{3n+2} &= p_{3n+1} + p_{3n} & \qquad q_{3n+2} &= q_{3n+1} + q_{3n} \end{aligned}$$

   with initial values $p_0 = p_1 = q_0 = q_2 = 1$, $q_1 = 0$, and $p_2 = 2$.

   (b) Now define $A_n = \int_0^1 \frac{x^n(x-1)^n}{n!} e^x \, dx$, $B_n = \int_0^1 \frac{x^{n+1}(x-1)^n}{n!} e^x \, dx$, $C_n = \int_0^1 \frac{x^n(x-1)^{n+1}}{n!} e^x \, dx$. Show that $A_n = -B_{n-1} - C_{n-1}$, $B_n = -2nA_n + C_{n-1}$, and $C_n = B_n - A_n$, and also that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = 0$. [Hint: For the first two, compute the derivatives of $\frac{1}{n!}x^n(x-1)^n e^x$ and $\frac{1}{n!}x^n(x-1)^{n+1}e^x$ and then integrate both sides from $x = 0$ to $x = 1$.]

   (c) With notation as in part (b), show that $A_n = -(p_{3n} - q_{3n}e)$, $B_n = p_{3n+1} - q_{3n+1}e$, and $C_n = p_{3n+2} - q_{3n+2}e$. [Hint: Show that A_n, B_n, C_n satisfy the same recurrences as the given combinations of p_n, q_n and also have the same initial values.]

   (d) Conclude that $\beta = \lim_{i\to\infty} p_i/q_i = e$, and from this fact deduce the continued fraction expansion of e.

Remark: This argument was originally given by Hermite, and is adapted from an article of H.A. Cohn.
