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1. Find all solutions in integers (if any) to the following linear Diophantine equations:

   (a) $22a + 17b = 19$.

   - Since $\gcd(22, 17) = 1$ divides 19, there is a solution.
   - Reducing both sides modulo 17 yields $5a \equiv 2 \pmod{17}$. Since $5^{-1} \equiv 7 \pmod{17}$, scaling by 7 yields $a \equiv 14 \pmod{17}$.
   - The general solution is $(a, b) = \boxed{(14 + 17k, -17 - 22k)}$ for $k \in \mathbb{Z}$.

   (b) $42a + 27b = 39$.

   - Since $\gcd(42, 27) = 3$ divides 39, there is a solution. Divide by 3 to get $14a + 9b = 13$.
   - Reducing both sides modulo 9 yields $5a \equiv -5 \pmod{9}$, which clearly has the solution $a \equiv -1 \pmod{9}$.
   - The general solution is $(a, b) = \boxed{(-1 + 9k, 3 - 14k)}$ for $k \in \mathbb{Z}$.

   (c) $3a + 7b + 16c = 8$.

   - Rewrite as $3(a + 2b + 5c) + b + c = 8$.
   - Now substitute $w = a + 2b + 5c$ to obtain the equation $3w + b + c = 8$, which we can easily solve as $b = -3w - c + 8$.
   - Substituting back yields $a = w - 2b - 5c = -16 - 3c + 7w$.
   - So the general solution is $(a, b, c) = \boxed{(-16 - 3c + 7w,\ -3w - c + 8,\ c)}$ for $c, w \in \mathbb{Z}$.

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2. Find all right triangles having one leg of length 40, and whose other two side lengths are integers.

   - Any such right triangle has legs of lengths $k(2st)$ and $k(s^2 - t^2)$, with hypotenuse $k(s^2 + t^2)$, where $s > t$ are unique positive integers of opposite parity and k is some unique positive integer.
   - If $40 = 2stk$, then $20 = stk$, so $(s, t, k) = (20, 1, 1)$ or $(10, 1, 2)$ or $(5, 4, 1)$ or $(5, 2, 2)$ or $(4, 1, 5)$ or $(2, 1, 10)$, yielding 40-399-401, 40-198-202, 9-40-41, 40-42-58, 40-75-85, and 30-40-50.
   - If $40 = k(s^2 - t^2)$, then k must be divisible by 8. Since $k \neq 40$ we see $k = 8$, and then $s^2 - t^2 = 5$ requires $s = 3$ and $t = 2$. This yields a 40-96-104 triangle.
   - Hence there are seven: $\boxed{\text{9-40-41}}$, $\boxed{\text{30-40-50}}$, $\boxed{\text{40-42-58}}$, $\boxed{\text{40-75-85}}$, $\boxed{\text{40-96-104}}$, $\boxed{\text{40-198-202}}$, and $\boxed{\text{40-399-401}}$.

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3. Byzantine Basketball is like regular basketball except that foul shots are worth a points instead of two points and field shots are worth b points instead of three points. Moreover, in Byzantine Basketball there are exactly 35 scores that never occur in a game, one of which is 58. What are a and b?

   - Clearly a and b must be relatively prime, or else there are infinitely many unattainable scores. Assume $a < b$.
   - By Sylvester's theorem, the number of unattainable scores is $\frac{1}{2}(a-1)(b-1)$, so $(a-1)(b-1) = 2 \cdot 35 = 2 \cdot 5 \cdot 7$.
   - Then $(a, b) = (2, 71)$, $(3, 36)$, $(6, 15)$, or $(8, 11)$. However, the middle two pairs are not relatively prime, and the first pair leaves 58 attainable.
   - Hence $(a, b) = \boxed{(8, 11) \text{ or } (11, 8)}$.

   Remark: This problem was on the 1971 Putnam exam.

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4. Compute the following things:

   (a) Show that 7/13 and 13/24 are adjacent in the Farey sequence of level 24. What are the next three terms after them?

   - We have $13 \cdot 13 - 7 \cdot 24 = 169 - 168 = 1$ so these terms are indeed consecutive in the Farey sequence of level $\max(13, 24) = 24$.
   - Using the recurrence, the next three terms are $\boxed{6/11,\ 11/20,\ 5/9}$.

   (b) Find all n such that exactly 2 terms appear between 7/13 and 13/24 in the Farey sequence of level n.

   - The first term that will appear between them is the mediant 20/37 in the Farey sequence of level 37.
   - Between 7/13 and 20/37 the next term will be 27/50, while between 20/37 and 13/24, the next intermediate term will be 33/61.
   - Thus, the first time there are two intermediate terms occurs with $n = 50$: the terms are 7/13, 27/50, 20/37, 13/24. The next term that will appear between any of these is 33/61, and so for $\boxed{n = 50, 51, \ldots, 60}$ there are exactly two intermediate terms.

   (c) List all the terms between 6/19 and 5/14 in the Farey sequence of level 19.

   - These terms are not consecutive because $5 \cdot 19 - 6 \cdot 14 = 11$ is not 1. To generate a term between them we try the mediant, which is $(6 + 5)/(19 + 14) = 11/33 = 1/3$.
   - We can then see 6/19 and 1/3 are consecutive since $19 - 6 \cdot 3 = 1$. Using the two-term recurrence to generate the next terms in this sequence produces 6/17 and then 5/14. Thus, the full list is $\boxed{6/19,\ 1/3,\ 6/17,\ 5/14}$. (We can check this by noting that $bc - ad = 1$ for each consecutive pair, and there are no missing terms because the mediants all have larger denominators.)
   - Alternatively, we could just have computed the next term after 6/19 directly, and then used the recurrence.

   (d) Find the three terms following 154/227 in the Farey sequence of level 2026.

   - By the above results, if 154/227 and c/d are consecutive terms, then $227c - 154d = 1$.
   - Solving this Diophantine equation using the Euclidean algorithm produces the solutions $(c, d) = (173 + 154k,\ 255 + 227k)$ for $k \in \mathbb{Z}$.
   - The larger the value of k is, the smaller the value of $\dfrac{c}{d} - \dfrac{154}{227} = \dfrac{1}{227d}$ will be. The largest possible value for k is $k = 7$, so the first term is $\dfrac{1251}{1844}$.
   - Now we can apply the two-term recursion to find the next terms, which are 1097/1617 and 943/1390.
   - Thus, the three terms are $\boxed{\dfrac{1251}{1844}, \dfrac{1097}{1617}, \dfrac{943}{1390}}$.

   (e) List all the terms between 1502/1801 and 1492/1789 in the Farey sequence of level 2026.

   - These terms are not consecutive since $1801 \cdot 1492 - 1502 \cdot 1789 = 14$ is not equal to 1.
   - Taking the mediant gives $(1502 + 1492)/(1801 + 1789) = 1497/1795$. This term is adjacent to 1502/1801 but not to 1492/1789. Taking the mediant again yields 427/512, which is adjacent to both.
   - To generate terms between 1502/1801 and 1497/1795, taking the mediant does not work since it does not reduce to give a smaller denominator. Instead what we can do is compute the next term in the sequence following 1502/1801 by solving $1801c - 1502b = 1$, which has solution $(c, d) = (643 + 1502k, 771 + 1801k)$ for $k \in \mathbb{Z}$.
   - Taking $k = 0$ yields the term 643/771. Then using the 2-term recurrence we can compute the next terms, which are 1070/1283 and 1497/1795.
   - Thus, the terms are $\boxed{\dfrac{1502}{1801}, \dfrac{643}{771}, \dfrac{1070}{1283}, \dfrac{1497}{1795}, \dfrac{427}{512}, \dfrac{1492}{1789}}$ since there are no terms in the Farey sequence of level 2024 that are between these.

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5. Find the continued fraction expansions for the following rational numbers:

   (a) 355/113.

   - Using the Euclidean algorithm we write

$$\begin{aligned} 355 &= 3 \cdot 113 + 16 \\ 113 &= 7 \cdot 16 + 1 \\ 16 &= 16 \cdot 1 \end{aligned}$$

   so $355/113 = \boxed{[3, 7, 16]}$.

   (b) 418/2021.

   - Using the Euclidean algorithm we write

$$\begin{aligned} 418 &= 0 \cdot 2021 + 418 \\ 2021 &= 4 \cdot 418 + 349 \\ 418 &= 1 \cdot 349 + 69 \\ 349 &= 5 \cdot 69 + 4 \\ 69 &= 17 \cdot 4 + 1 \\ 4 &= 4 \cdot 1 \end{aligned}$$

   so $418/2021 = \boxed{[0, 4, 1, 5, 17, 4]}$.

   (c) 13579/2468.

   - Using the Euclidean algorithm we write

$$\begin{aligned} 13579 &= 5 \cdot 2468 + 1239 \\ 2468 &= 1 \cdot 1239 + 1229 \\ 1239 &= 1 \cdot 1229 + 10 \\ 1229 &= 122 \cdot 10 + 9 \\ 10 &= 1 \cdot 9 + 1 \\ 9 &= 9 \cdot 1 \end{aligned}$$

   so $13579/2468 = \boxed{[5, 1, 1, 122, 1, 9]}$.

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6. Prove that the area of any right triangle with integer side lengths is always divisible by 6, but not necessarily any integer greater than 6.

   - From our characterization we can see that the area of a Pythagorean right triangle is of the form $\frac{1}{2} \cdot 2kst \cdot k(s^2 - t^2) = k^2 st(s^2 - t^2)$ where s and t are relatively prime integers of opposite parity.
   - Since one of s, t is even, the area is also even.
   - If neither s nor t is divisible by 3, then they are either 1 or 2 mod 3, so their squares are both 1 mod 3. But then $s^2 - t^2$ is divisible by 3. Thus, the area is also divisible by 3.
   - Since the area is divisible by both 2 and 3, it is a multiple of 6.
   - The 3-4-5 triangle has area 6, so clearly the area need not be divisible by anything greater than 6.

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7. The goal of this problem is to discuss the Weierstrass substitution $t = \tan(\theta/2)$. Recall the trigonometric identities $\tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$. Let $t = \tan(\theta/2)$.

   (a) Show that $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$, and $d\theta = \dfrac{2dt}{1+t^2}$.

   - As given above, we have $t = \dfrac{\sin\theta}{1+\cos\theta}$ and $t = \dfrac{1-\cos\theta}{\sin\theta}$, yielding the linear system $\sin\theta - t\cos\theta = t$ and $t\sin\theta + \cos\theta = 1$. Multiplying the second equation by t and adding yields $(1+t^2)\sin\theta = 2t$ so $\sin\theta = \dfrac{2t}{1+t^2}$ and back-substituting yields $\cos\theta = \dfrac{1-t^2}{1+t^2}$.
   - Finally, since $\theta = 2\tan^{-1} t$, differentiating yields $d\theta = \dfrac{2dt}{1+t^2}$.

   (b) Use the Weierstrass substitution to compute the indefinite integral $\displaystyle\int \frac{1}{5+3\cos\theta}\,d\theta$.

   - By (a), with $t = \tan(\theta/2)$ we have $\cos\theta = \dfrac{1-t^2}{1+t^2}$ and $d\theta = \dfrac{2}{1+t^2}dt$. Substituting yields $\int \dfrac{1}{5+3\cos\theta}\,d\theta = \displaystyle\int \frac{1}{5+3\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{2}{5(1+t^2)+3(1-t^2)}\,dt = \int \frac{1}{4+t^2}\,dt = \frac{1}{2}\tan^{-1}\frac{t}{2} + C.$
   - Thus, the original integral is $\boxed{\dfrac{1}{2}\tan^{-1}\left(\dfrac{1}{2}\tan(\theta/2)\right) + C}$.

   (c) Use the Weierstrass substitution to compute the definite integral $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{4-\sin\theta}\,d\theta$.

   - Substituting $t = \tan(\theta/2)$ yields $\int_{-\pi/2}^{\pi/2} \dfrac{1}{4-\sin\theta}\,d\theta = \int_{-1}^{1} \dfrac{1}{4-\dfrac{2t}{1+t^2}}\cdot\dfrac{2}{1+t^2}\,dt = \int_{-1}^{1} \dfrac{1}{2t^2-t+2}\,dt =$ $\int_{-1}^{1} \dfrac{1}{(4t-1)^2/15+1}\,dt = \dfrac{2}{\sqrt{15}}\tan^{-1}\dfrac{4t-1}{\sqrt{15}}\Big|_{t=-1}^{1} = \boxed{\dfrac{\pi}{\sqrt{15}}}$.

   Remark: As can be seen from (b) and (c), the Weierstrass substitution allows evaluation of any integral that is a rational function of $\sin\theta$ and $\cos\theta$.

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8. Find the smallest positive integer n such that for all integers m with $0 < m < 2026$, there exists an integer k with $\dfrac{m}{2026} < \dfrac{k}{n} < \dfrac{m+1}{2027}$. (Make sure to prove that your value is the smallest possible.)

   - The answer is $n = 4053$. From our results on Farey fractions, we know the mediant $\dfrac{2m+1}{4053}$ is always between $\dfrac{m}{2026}$ and $\dfrac{m+1}{2027}$. Thus, $n = 4053$ will work.
   - On the other hand, if we take $m = 2025$, then we require $\dfrac{2025}{2026} < \dfrac{k}{n} < \dfrac{2026}{2027}$. Since $2026\cdot 2026 - 2025\cdot 2027 = 1$, the terms $\dfrac{2025}{2026}$ and $\dfrac{2026}{2027}$ are consecutive in the Farey sequence, so the next term that appears between then is $\dfrac{4051}{4053}$. This means $n \geq 4053$, and so indeed $n = 4053$ is the smallest possible n.

   Remark: This is a variation of problem B1 from the 1993 Putnam exam.

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9. For each Farey fraction a/b, define $\mathcal{C}(a/b)$ to be the circle in the upper-half of the Cartesian plane of radius $r_{a/b} = 1/(2b^2)$ that is tangent to the x-axis at the point $(a/b, 0)$. These circles are called the Ford circles.

   (a) If a/b and c/d are adjacent entries in some Farey sequence, prove that the circles $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are tangent.

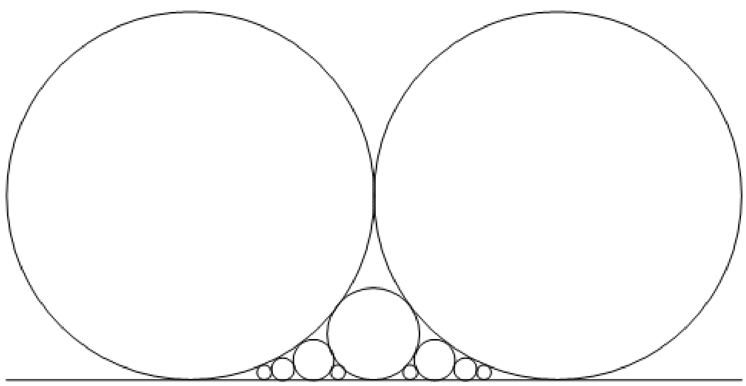
   - By our results on the Farey sequences, we know that a/b and c/d are adjacent if and only if $bc - ad = 1$.
   - The centers of the two circles are $(a/b, r_{a/b})$ and $(c/d, r_{c/d})$, so the tangency condition is equivalent to saying that $(r_{c/d} - r_{a/b})^2 + (a/b - c/d)^2 = (r_{c/d} + r_{a/b})^2$, which is equivalent to $(a/b - c/d)^2 = 4r_{a/b}r_{c/d}$.
   - Since $r_{a/b} = 1/(2b^2)$ and $r_{c/d} = 1/(2d^2)$, both sides are equal to $\frac{1}{b^2d^2}$, and so the circles are indeed tangent.

   (b) If a/b and c/d are not adjacent in any Farey sequence, prove that the interiors of the circles $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are disjoint.

   - Using the calculation in (a), we can see that the distance between the centers is equal to $(r_{c/d} - r_{a/b})^2 + (a/b - c/d)^2$, which will exceed $r_{c/d} + r_{a/b}$ whenever $|bc - ad| > 1$. Thus, if the terms are not adjacent, the interiors are disjoint.

   (c) Draw (you should probably use a computer) the 11 Ford circles corresponding to the Farey sequence of level 5.

   - Here's a plot:



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10. [Challenge] Let a, b, c be pairwise relatively prime positive integers. Show that $2abc - ab - bc - ca$ is the largest integer that cannot be expressed in the form $xbc + yca + zab$ for nonnegative integers x, y, z. [Hint: Any integer is congruent modulo a to precisely one of 0, bc, $2bc$, ... , $(a-1)bc$.]

- By Sylvester's theorem, the largest integer that cannot be written in the form $pb + qc$ is $bc - b - c$. Thus, multiplying by a shows that any multiple of a exceeding $a(bc - b - c) = abc - ab - ac$ can be written in the form $pab + qac$.
- But since bc is relatively prime to a, we see that the a integers 0, bc, ... , $(a-1)bc$ represent all of the residue classes modulo a.
- Therefore, any integer N exceeding $a(bc - b - c) + (a-1)bc = 2abc - ab - bc - ca$ is congruent to one of those a integers, and the difference $N - kbc$ is a multiple of a larger than $a(bc - b - c)$, hence is of the form $pab + qac$ for some nonnegative p, q by the above. This means $N = kbc + pab + qac$ for nonnegative k, p, q as required.
- However, if $N = 2ab - ab - bc - ca$ could be written as $xbc + yca + zab$ then modulo a we have $-bc \equiv xbc \pmod{a}$ so that $x \equiv -1 \pmod{a}$ and thus $x \geq a - 1$. Likewise we have $y \geq b - 1$ and $z \geq c - 1$. But then $xbc + ycz + zab \geq 3abc - ab - bc - ca > N$, contradiction.

Remark: This is a Frobenius coin problem for the integers ab, bc, ca and was problem 3 from the 1983 IMO.
