E. Dummit's Math 3527 ~ Number Theory 1, Fall 2026 ~ Homework 1, due Fri Sep 18th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem. **Use of generative AI in any manner is not allowed on this or any other course assignments.**

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Match the erroneous proofs (a)-(f) with the reasons (1)-(5) they are not valid inductive proofs of the claims. Some reasons may be used more than once and others not at all.

   Reasons:

   (1) The proof does not actually check that the base case is correct.
   (2) The proof of the inductive step shows that the claimed result implies a true statement instead of proving the claimed result.
   (3) The proof of the inductive step assumes more base cases than are actually checked.
   (4) The proof of the inductive step shows that the result for $n+1$ implies the result for n, rather than the other way around.
   (5) The inductive step makes an invalid assumption about n.

   (a) Proposition: For every positive integer n, $1+2+4+\cdots+2^n = 2n+1$.
   Proof: Induction on n. The base case $n=1$ follows because $1+2 = 2\cdot 1+1$. For the inductive step, we want to show that $1+2+4+\cdots+2^{n+1} = 2^{n+2}-1$. Multiplying both sides by 0 yields $0=0$, which is a true statement. Therefore the result holds by induction.

   (b) Proposition: If $a_1 = 3$ and $a_{n+1} = 3a_n + 2$ for all $n \geq 1$, then $a_n = 3^n - 1$ for all n.
   Proof: Induction on n. The base case $n=1$ is trivial. For the inductive step, suppose $a_n = 3^n - 1$. Then $a_{n+1} = 3a_n + 2 = 3(3^n - 1) + 2 = 3^{n+1} - 1$ as required.

   (c) Proposition: All horses are the same color.
   Proof: Induction on n, the number of horses. The base case $n=1$ is trivial. For the inductive step, suppose that any $n+1$ horses are the same color. Ignoring the last horse yields means that we need to show that n horses are the same color, which is true by the induction hypothesis. Therefore the result holds by induction.

   (d) Proposition: For every positive integer n, $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$.
   Proof: Induction on n. The base case $n=1$ follows because $1 = \frac{1}{2}(1)(2)$. For the inductive step, suppose $1+2+3+\cdots+n+(n+1) = \frac{1}{2}(n+1)(n+2)$. Subtracting $n+1$ from both sides yields $1+2+3+\cdots+n = \frac{1}{2}(n+1)(n+2)-(n+1) = \frac{1}{2}n(n+1)$, as required. Thus the result holds by induction.

   (e) Proposition: If $a_1 = 2$, and $a_{n+1} = 4a_n - 4a_{n-1}$ for all $n \geq 1$, then $a_n = 2^n$ for all n.
   Proof: Strong induction on n. The base case $n=1$ follows since $a_1 = 2 = 2^1$. For the inductive step, suppose $a_k = 2^k$ for all $k \leq n$. Then $a_{n+1} = 4a_n - 4a_{n-1} = 4\cdot 2^n - 4\cdot 2^{n-1} = 4\cdot 2^n - 2\cdot 2^n = 2\cdot 2^n = 2^{n+1}$ as required.

   (f) Proposition: For all positive integers n, $2^n > n^2$.
   Proof: Induction on n. The base case $n=1$ follows since $2>1$. For the inductive step, suppose $2^n > n^2$. Then $2^{n+1} = 2\cdot 2^n > 2n^2 > n^2+n^2 \geq n^2+3n > n^2+2n+1 = (n+1)^2$ since $n \geq 3$.

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2. The goal of this problem is to explore some methods for finding greatest common divisors.

   (a) Find all divisors of 20 and of 28, list the common divisors, and use the list to find $\gcd(20, 28)$.

   (b) Find all divisors of 33 and of 90, list the common divisors, and use the list to find $\gcd(33, 90)$.

   (c) Claim: Starting with any pair of positive integers (a, b), repeatedly subtract the smaller number in the pair from the larger, until one of the numbers reaches zero: then the other number is $\gcd(a, b)$.
   Test the claim using the pairs $(20, 28)$ and $(33, 90)$: does it give the correct gcd?

   (d) Which method, of the ones given in (a)/(b) and in (c), would be faster on two 10-digit numbers?

   Remark: In fact, the method described in (c), using iterated subtraction, was the original formulation of the Euclidean algorithm. (The version we discuss in lecture is faster, but requires computing remainders.)

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Prove the following:

   (a) For every integer $n \geq 2$, it is true that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \frac{1}{3}n(n-1)(n+1)$.

   (b) Let $a_0 = 0$, $a_1 = 1$, and, for $n \geq 0$, let $a_{n+2} = 5a_{n+1} - 6a_n$. Prove that $a_n = 3^n - 2^n$ for all $n \geq 0$.

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4. Prove the following basic properties of divisibility (note that some of these properties are referred to, but not proven, in the course notes; you are expected to give the details of the proof!):

   (a) If a, b are integers, show that $a|b$ if and only if $a|(-b)$.

   (b) If a, b, m are integers with $m \neq 0$, show that $a|b$ if and only if $(ma)|(mb)$.

   (c) If a, b, c, x, y are integers such that $a|b$ and $a|c$, show that $a|(xb + yc)$.

   (d) If a, b are integers, show that a, b and $a, a - b$ have the same set of common divisors and the same gcd.

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5. The Fibonacci numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. The first few terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

   (a) Prove that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ for every positive integer n.

   (b) Prove that $F_{n+1}^2 - F_n F_{n+2} = (-1)^n$ for every positive integer n.

   (c) Prove that $F_{2n+1} = F_{n+1}^2 + F_n^2$ and $F_{2n+2} = F_{n+1}(F_{n+2} + F_n)$ for all $n \geq 1$. [Hint: Show both together by induction.]

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6. The goal of this problem is to study a few more miscellaneous properties of Fibonacci numbers, as defined in problem 5. (By convention, we also take $F_0 = 0$.)

   (a) Find $\gcd(F_5, F_{10})$, $\gcd(F_6, F_9)$, $\gcd(F_6, F_{12})$, and $\gcd(F_{12}, F_{13})$.

   (b) Show that $F_{n+k+1} = F_{k+1}F_{n+1} + F_k F_n$, for any nonnegative n and k. [Hint: Strong induction on k.]

   (c) Show that $F_a | F_{na}$ for all positive integers n. [Hint: Use (b).]

   (d) Show that F_n and F_{n+1} are relatively prime for all n.

   Remark: It can in fact be shown using the results in (b), (c), and (d) that the gcd of any two Fibonacci numbers is another Fibonacci number, and more specifically that $\gcd(F_a, F_b) = F_{\gcd(a,b)}$.
