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1. Parts (a), (b), and (c) were worth 3 points, and each item in (d) was worth 1 point.

   (a) Since $i = e^{i\pi/2}$ the fourth roots of $i$ are $e^{i\pi/8+2ki\pi/4} = \boxed{e^{i\pi/8}, e^{5i\pi/8}, e^{9i\pi/8}, e^{13i\pi/8}}$.

   (b) Since $9i = 9e^{i\pi/2}$ we have $\log(9i) = \boxed{\ln 9 + i\pi/2 + 2k\pi i}$ for integers $k$.

   (c) We have $(9i)^i = e^{i\log(9i)} = e^{i[\ln 9 + i\pi/2 + 2k\pi i]} = \boxed{e^{-\pi/2-2k\pi}(\cos\ln 9 + i\sin\ln 9)}$ for integers $k$.

   (d) $\boxed{\text{False}}$, $\boxed{\text{True}}$, $\boxed{\text{True}}$, $\boxed{\text{False}}$, $\boxed{\text{False}}$: $R$ is not closed since it does not contain its boundary, $R$ is bounded since it is contained in $|z| < 10$, $R$ is connected since a path can be drawn in $R$ from any point to any other, $R$ is not simply connected as (e.g.,) the circle $|z| = 5$ is not homotopic to the trivial path, and the principal logarithm is discontinuous on the positive real axis (part of which is in $R$).

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2. Each part was worth 6 points.

   (a) As $f(z) = -iz + z^2\overline{z}^2$ we see $\partial f/\partial\overline{z} = 2z^2\overline{z}$ which is nonzero except at $z = 0$, so the derivative $f'$ cannot exist anywhere except at $z = 0$. To compute $f'(0)$ we use the limit definition of the derivative to see $f'(0) = \lim_{h\to 0}\dfrac{f(h) - f(0)}{h - 0} = \lim_{h\to 0}\dfrac{-ih + h^2\overline{h}^2}{h} = \lim_{h\to 0}(-i + h\overline{h}^2) = -i$. (Note that we cannot just compute $\partial f/\partial z$ because $\partial f/\partial z = f'$ only when $f$ is holomorphic on a region.)

   (b) For $f(x+iy) = u(x,y) + iv(x,y)$ we have $f^*(x+iy) = u(x,-y) - iv(x,-y)$, so $\frac{\partial f^*}{\partial\overline{z}}(x+iy) = \frac{1}{2}\left[\frac{\partial f^*}{\partial x} + i\frac{\partial f^*}{\partial y}\right](x+iy) = \frac{1}{2}[u_x(x,-y) - iv_x(x,-y) + i(-u_y(x,-y) + iv_y(x,-y))] = \frac{1}{2}[f_x(x,-y) - if_y(x,-y)] = \overline{\frac{\partial f}{\partial\overline{z}}(x-iy)}$. So for $x+iy \in R$ we see that $f^*$ is holomorphic at $x - iy$, so since $f$ is holomorphic on $R$, $f^*$ is holomorphic on $\overline{R}$.
   Alternatively, letting $g(z) = \overline{z}$, we see $f^*(z) = g(f(g(z)))$ so $\frac{\partial f^*}{\partial\overline{z}}(z_0) = \frac{\partial g}{\partial\overline{z}}(f(g(z_0))\cdot\frac{\partial f}{\partial\overline{z}}(g(z_0))\cdot\frac{\partial g}{\partial\overline{z}}(z_0) = \frac{\partial f}{\partial\overline{z}}(\overline{z_0})$ so for $z_0 \in \overline{R}$ this evaluates to zero since $f$ is holomorphic at $\overline{z_0} \in R$.
   Alternatively, since holomorphic functions are analytic, we can write $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ with a positive radius of convergence, for any $z_0$ in $R$. Then $f^*(z) = \overline{f(\overline{z})} = \overline{\sum_{n=0}^{\infty} a_n(\overline{z}-z_0)^n} = \sum_{n=0}^{\infty}\overline{a_n}(z-\overline{z_0})^n$ which has the same radius of convergence as the original series since $\limsup_{n\to\infty}|a_n|^{1/n} = \limsup_{n\to\infty}|\overline{a_n}|^{1/n}$. In particular, the new series is differentiable at $z = \overline{z_0}$, so $f^*$ is holomorphic on $\overline{R}$.

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3. Parts (a) and (b) were worth 3 points, part (c) was worth 4 points, and part (d) was worth 2 points.

   (a) We have $\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty} n^{1/n}(n+1)^{1/n}/2^{1/n} = 1$ so the radius is $1/1 = \boxed{1}$ and the disc is $\boxed{|z| < 1}$.

   (b) We can integrate term by term to see $F(z) = C + \sum_{n=0}^{\infty}\dfrac{n}{2}z^{n+1}$, and setting $z = 0$ gives $C = 2025$, so $F(z) = 2025 + \sum_{n=0}^{\infty}\dfrac{n}{2}z^{n+1} = \boxed{2025 + \dfrac{1}{2}z^2 + z^3 + \dfrac{3}{2}z^4 + \cdots}$.

   (c) Since $f(z) = z(1 + 3z + 6z^2 + 10z^3)$ we see $1/f(z) = z^{-1}(1 + 3z + 6z^2 + 10z^3)^{-1}$ so we need the second term out to $z^2$. If $1 + 3z + 6z^2 + 10z^3 + \cdots$ has inverse $b_0 + b_1 z + b_2 z^2 + \cdots$, multiplying out $(1+3z+6z^2+\cdots)(b_0+b_1z+b_2z^2+\cdots) = b_0 + (3b_0+b_1) + (6b_0+3b_1+b_2) + \cdots$ and solving yields $b_0 = 1$, $3b_0 + b_1 = 0$ so $b_1 = -3$, $6b_0 + 3b_1 + b_2 = 0$ so $b_2 = -6b_0 - 3b_1 = 3$. Then $f(z) = z^{-1}(1 - 3z + 3z^2 + \cdots) = \boxed{z^{-1} - 3 + 3z + \cdots}$. (In fact, one can show $1/f(z) = z^{-1} - 3 + 3z - z^2$ exactly.)

   (d) Since $f(z)$ is holomorphic inside its disc of convergence, which is a simply connected region, by Cauchy's integral theorem or our results on integrating power series we see that $\int_\gamma f(z)\,dz = \boxed{0}$ for any such $\gamma$.

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4. Each part was worth 4 points.

   (a) For $f(z) = |z|^2$ we see $f(3e^{it}) = \left|3e^{it}\right|^2 = 9$ and $\gamma'(t) = 3ie^{it}$, so we get $\int_0^{\pi} 9 \cdot 3ie^{it}\,dt = 27e^{it}|_{t=0}^{\pi} = \boxed{-54}$.

   (b) We have a parametrization $\gamma(t) = i + t(3-5i) = 3t + (1-5t)i$ for $0 \le t \le 1$. Then $f(z) = [\mathrm{Re}(z)]^2$ we see $f(\gamma(t)) = (3t)^2$ and $\gamma'(t) = 3-5i$, so we get $\int_0^1 (3t)^2(3-5i)\,dt = 3t^3(3-5i)|_{t=0}^{1} = 3(3-5i) = \boxed{9-15i}$.

   (c) Since $f(z) = 3z^2$ is holomorphic and has an antiderivative $F(z) = z^3$, the start point is $\gamma(0) = 0$ and the end point is $\gamma(1) = e$, by the fundamental theorem of calculus / independence of path we see that $\int_\gamma f(z)\,dz = F(e) - F(0) = \boxed{e^3}$.

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5. Each part was worth 4 points.

   (a) Since $e^{2z} = \sum_{n=0}^{\infty} \dfrac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \dfrac{2^n}{n!} z^n$ we see $\dfrac{e^{2z}}{z^4} = \boxed{\sum_{n=0}^{\infty} \dfrac{2^n}{n!} z^{n-4} = z^{-4} + 2z^{-3} + 2z^{-2} + \dfrac{4}{3}z^{-1} + \dfrac{2}{3} + \cdots}$.

   (b) Since the radius of convergence for the power series is $\infty$, and the contour winds once counterclockwise around the origin, by our results on integrating Laurent series we see that $\int_\gamma f(z)\,dz = 2\pi i W_\gamma(0) a_{-1} = 2\pi i \cdot 1 \cdot \frac{4}{3} = \boxed{\frac{8\pi i}{3}}$.

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6. Parts (a) and (c) were each worth 3 points, part (b) was worth 2 points, and part (d) was worth 4 points.

   (a) By decomposing the contour into simple pieces, or by drawing a ray from the given point out to $\infty$ in any direction, the winding number around 0 is $\boxed{0}$, the winding number around $i$ is $\boxed{+2}$, and the winding number around $3i$ is $\boxed{-1}$.

   (b) By the definition of winding number, we see $\int_\gamma \dfrac{1}{z}\,dz = 2\pi i W_\gamma(0) = \boxed{0}$.

   (c) By Cauchy's integral formula with $f(z) = e^z$ and $z_0 = i$, $\int_\gamma \dfrac{e^z}{z-i}\,dz = 2\pi i W_\gamma(i) f(i) = 2\pi i \cdot 2 \cdot e^i = \boxed{4\pi i e^i}$.

   (d) We have the partial fraction decomposition $\dfrac{1}{z^2+1} = \dfrac{i/2}{z+i} - \dfrac{i/2}{z-i}$. Then by the definition of winding number (twice) we see $\int_\gamma \dfrac{1}{z^2+1}\,dz = \dfrac{i}{2} \cdot 2\pi i W_\gamma(-i) - \dfrac{i}{2} \cdot 2\pi i W_\gamma(i) = 2\pi W_\gamma(i) - 2\pi W_\gamma(-i) = \boxed{2\pi}$.

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7. Part (a) was worth 6 points and part (b) was worth 4 points.

   (a) First we show that the series has radius of convergence $\infty$. This follows either by the ratio test, since $|a_{n+1}/a_n| = \frac{1}{(3n+3)(3n+2)(3n+1)} \to 0$ as $n \to \infty$, or directly by noting $\lim_{n\to\infty} |a_n|^{1/n} \le \lim_{n\to\infty} \frac{1}{(3n!)^{1/n}} = \lim_{n\to\infty} \frac{1}{(n/e)\sqrt{2\pi n}^{1/n}} = 0$ via Stirling's approximation. Then by our results on power series, $f(z)$ is holomorphic inside its radius of convergence, hence on all of $\mathbb{C}$. For the second part we can differentiate $f(z) = 1 + \dfrac{z^3}{3!} + \dfrac{z^6}{6!} + \dfrac{z^9}{9!} + \cdots$ term by term to see $f'(z) = \dfrac{z^2}{2!} + \dfrac{z^5}{5!} + \dfrac{z^8}{8!} + \cdots$ and $f''(z) = z + \dfrac{z^4}{4!} + \dfrac{z^7}{7!} + \cdots$: then $f''(z) + f'(z) + f(z) = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = e^z$.

   (b) The differentiation-via-integration formula says that for a holomorphic function $g$ on a simply-connected region $R$ with counterclockwise boundary $\gamma$, we have $g^{(n)}(z_0) = \dfrac{n!}{2\pi i} \int_\gamma \dfrac{g(z)}{(z-z_0)^{n+1}}\,dz$. Applying the formula with $g = f$ and $n = 1$ shows that $\int_\gamma \dfrac{f(z)}{(z-z_0)^2}\,dz = 2\pi i f'(z_0)$, and applying the formula with $g = f'$ and $n = 0$ shows that $\int_\gamma \dfrac{f'(z)}{z-z_0}\,dz = 2\pi i f'(z_0)$ (alternatively this follows from Cauchy's integral formula applied to $f'$): thus, the expressions are equal.

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