1. (a) $(1+5i)/13$.
   (b) 2^{400}.
   (c) $\sqrt{13}$.
   (d) $z = 2i$, $\sqrt{3}-i$, $-\sqrt{3}-i$.
   (e) $z = 2i$, $-3i$.
   (f) $z = \pm\sqrt{45}$, $\pm\sqrt{45}i$.
   (g) $z = \frac{1}{4}(\ln 4 + 2\pi i k)$, k integral.
   (h) $e^{iz} = 3, -1/3$ so $z = -i\ln 3 + 2\pi k$ or $z = i\ln 3 + \pi + 2\pi k$, k integral.
   (i) $\ln(2)/2 + i\pi/4 + 2\pi i k$, k integral.
   (j) $i\pi + 2\pi i k$, k integral.
   (k) $e^{-\pi/2-2\pi k}$, k integral.
   (l) $e^{-i\ln 2+\pi/2+4\pi k} = e^{\pi/2+4\pi k}(\cos\ln 2 - i\sin\ln 2)$, k integral.

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2. Using the series definition of e^z gives $\overline{e^z} = \overline{\sum_{n=0}^{\infty} z^n/n!} = \sum_{n=0}^{\infty} \overline{z}^n/n! = e^{\overline{z}} = e^{1/z}$ since $\overline{z} = 1/z$ for $|z| = 1$.

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3. R_1 is open, not closed, bounded, connected, not simply connected.
   R_2 is not open, closed, not bounded, connected, simply connected.

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4. (a) $\partial f/\partial z = 3z^2$, $\partial f/\partial \overline{z} = i$, f is not holomorphic on any region.
   (b) $f = e^{(z+\overline{z})/2}$ so $\partial f/\partial z = \partial f/\partial \overline{z} = \frac{1}{2}e^{(z+\overline{z})/2}$, f is not holomorphic on any region.
   (c) $\partial f/\partial z = e^{z\,\mathrm{Log}(z)}(\mathrm{Log}(z)+1)$, $\partial f/\partial \overline{z} = 0$, f is holomorphic on $\mathbb{C}\backslash[0,\infty)$ because of the branch cut for $\mathrm{Log}(z)$.

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5. $f(z) = z\overline{z}$ so $\partial f/\partial \overline{z} = z$ which is zero only at $z = 0$. At $z = 0$ we have $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h-0} = \lim_{h\to 0} \frac{h\overline{h}}{h} = \lim_{h\to 0} \overline{h} = 0$ so the derivative exists and is zero.

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6. For $u = e^{-2xy}\cos(x^2-y^2)$, $v = e^{-2xy}\sin(x^2-y^2)$ we have $\partial u/\partial x = -2ye^{-2xy}\cos(x^2-y^2) - 2xe^{-2xy}\sin(x^2-y^2) = \partial v/\partial y$ and $\partial u/\partial y = -2xe^{-2xy}\cos(x^2-y^2) + 2ye^{-2xy}\sin(x^2-y^2) = -\partial v/\partial x$. So f satisfies the Cauchy-Riemann equations hence is holomorphic. Alternatively, observe $f(z) = e^{iz^2}$ which is holomorphic with derivative $f'(z) = 2ize^{iz^2}$.

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7. (a) $\lim_{n\to\infty} |3^n|^{1/n} = 3$ so radius is $1/3$, disc is $|z| < 1/3$.
   (b) $\lim_{n\to\infty} |1/2^n|^{1/n} = 1/2$ so radius is 2, disc is $|z-i| < 2$.
   (c) $\sum_{n=0}^{\infty} 2^n z^n$.
   (d) $\sum_{n=0}^{\infty} (-1)^{n+1} 2^n (z-1)^n$.
   (e) $\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} z^{2n-3}$.
   (f) $1 - z - z^2 + z^3 + \cdots$
   (g) $z^{-3} - z^{-2} + 1 + \cdots$

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8. We have $\lim_{n\to\infty} |1/n|^{1/n} = 1$ so f has radius of convergence 1. Inside the radius we can differentiate term by term, yielding $f'(z) = \sum_{n=1}^{\infty} z^{n-1} = 1 + z + z^2 + \cdots = 1/(1-z)$, which holds for $|z| < 1$ as claimed.

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9. For $z = x + iy$ using $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ we see that $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{-y+ix} + e^{y-ix})$ and therefore $|\cos z|^2 = (\cos z)(\overline{\cos z}) = \frac{1}{4}(e^{-y+ix} + e^{y-ix})(e^{-y-ix} + e^{y+ix}) = \frac{1}{4}(e^{-2y} + e^{-2ix} + e^{2ix} + e^{2y}) = \frac{1}{2}(\cosh 2y + \cos 2x)$.

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10. Note that for line integrals of a non-holomorphic function, our only option is to set them up with a parametrization. For line integrals of a holomorphic function on a non-closed contour, we usually want to use the fundamental theorem of calculus. For line integrals on closed contours, we usually use a power series expansion or Cauchy's integral formula.

    (a) $I = \int_0^1 t^2(2t + 3t^2 i)\, dt = 1/2 + 3/5i$.

    (b) Antiderivative is $F(z) = z^3/3$ so $I = F(\gamma(1)) - F(\gamma(0)) = (3+6i)^3/3$.

    (c) Function is holomorphic on the interior of the closed contour so $I = 0$ by Cauchy's theorem/formula.

    (d) Antiderivative is $F(z) = z^4 + 2\mathrm{Log}(z)$ so $I = F(1+i) - F(1) = (1+i)^4 + 2\mathrm{Log}(1+i) - 1 = -5 + \ln 2 + i\pi/2$.

    (e) $I = \int_0^{\pi/2} \frac{1}{3e^{it}}(3ie^{it})\, dt = i\pi/2$.

    (f) $I = \int_0^{2\pi} \frac{1}{3e^{it}}(3ie^{it})\, dt = 2\pi i$, or $2\pi i$ directly by Cauchy's integral formula.

    (g) $I = 0$ by deforming the contour to a point.

    (h) For $f(z) = e^z$ Cauchy's integral formula gives $I = 2\pi i W_\gamma(0) f(0) = 2\pi i$.

    (i) For $f(z) = e^z$ Cauchy's integral formula gives $I = 2\pi i W_\gamma(1) f(1) = 2\pi e i$.

    (j) For $f(z) = z^2$ Cauchy's integral formula gives $I = 2\pi i W_\gamma(1) f(1) = 2\pi i$.

    (k) For $f(z) = e^z + \sin(2z)$ Cauchy's integral formula gives $I = 2\pi i W_\gamma(1) f(1) = 2\pi i(e + \sin 2)$.

    (l) As $\cos z/z^5 = z^{-5} - \frac{1}{2}z^{-3} + \frac{1}{24}z^{-1} - \frac{1}{720}z + \cdots$, the power series formula gives $I = 2\pi i W_\gamma(0) a_{-1} = \frac{2\pi i}{24}$.

    (m) By deforming the contour this is the same as the previous integral, which was $\frac{2\pi i}{24}$.

    (n) For $f(z) = e^z$ Cauchy's integral formula gives $I = 2\pi i W_\gamma(5) f(5) = 0$ since 5 is not in the circle.

    (o) For $f(z) = 1/(z+2i)$, $z_0 = 2i$ Cauchy's integral formula gives $I = 2\pi i W_\gamma(2i) f(2i) = \pi/2$ since f is holomorphic inside the circle as it doesn't contain $-2i$.

    (p) For $f(z) = 1/(z-5)$, $z_0 = 0$ Cauchy's integral formula gives $I = 2\pi i W_\gamma(0) f(0) = -2\pi i/5$ since f is holomorphic inside the square as it doesn't contain 5.

    (q) By partial fractions $f(z) = \frac{1/5}{z-5} - \frac{1/5}{z}$ and by Cauchy on each term we see $I = (2\pi i/5) - (2\pi i/5) = 0$.

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11. (a) The winding numbers are $+2$, $+1$, $+1$, -1, -1, $+1$, 0.

    (b) $\int_\gamma \frac{1}{z}\, dz = 2\pi i \cdot W_\gamma(0) = 4\pi i$ by the definition of winding number.

    (c) $\int_\gamma \frac{1}{z-3}\, dz = 2\pi i \cdot W_\gamma(3) = -2\pi i$ by the definition of winding number.

    (d) $\int_\gamma \frac{e^z}{z-2}\, dz = 2\pi i \cdot W_\gamma(2) f(2) = 2\pi i e^2$ by Cauchy's integral formula.

    (e) $\int_\gamma \frac{e^z}{z-4}\, dz = 2\pi i \cdot W_\gamma(4) f(4) = -2\pi i e^4$ by Cauchy's integral formula.

    (f) $\int_\gamma \frac{e^z}{z^2-6z+8}\, dz = \frac{1}{2}\int_\gamma \frac{e^z}{z-4}\, dz - \frac{1}{2}\int_\gamma \frac{e^z}{z-2}\, dz = -\pi i e^4 - \pi i e^2$ by partial fractions and the above.

    (g) $\frac{\cos z}{z^3} = z^{-3} - \frac{1}{2}z^{-1} + \frac{1}{6}z + \cdots$ so $\int_\gamma \frac{\cos z}{z^3}\, dz = 2\pi i \cdot W_\gamma(0) \cdot a_{-1} = -2\pi i$ via series expansion.

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12. By the differentiation-via-integration formula we have $\int_\gamma \frac{f(z)}{(z-z_0)^2}\, dz = f'(z_0)$, so $f'(z_0)$ is identically zero on $\mathbb{C}$. As shown in class (or as follows by noting $f(b) - f(a) = \int_{\gamma'} f'(z)\, dz = 0$ on any contour γ' from a to b) this means f is constant.
