Math 4555: Complex Variables

Midterm Exam (Instructor: Dummit)

October 29th, 2025

NAME (please print legibly): ______________________________

- **Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!**
- **You may appeal to results covered at any point in the course, but please make clear what results you are using. $\boxed{\textbf{Box}}$ all final numerical answers.**
- **You are responsible for checking that this exam has all 9 pages.**
- **You are allowed a calculator and a 1-page note sheet. Time limit: $\boxed{\textbf{100 minutes}}$.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	14	
2	12	
3	12	
4	12	
5	8	
6	12	
7	10	
TOTAL	80	

1. (14 points) Calculate the requested things:

(a) Find all complex z with $z^4 = i$.

(b) Find $\log(9i)$, where log denotes the multivalued complex logarithm.

(c) Find all complex values of the expression $(9i)^i$.

(d) Let R be the region consisting of all complex z such that $2 < |z| < 10$. Identify each statement as true or false (each answer is worth independent points):

True **False** The region R is closed.

True **False** The region R is bounded.

True **False** The region R is connected.

True **False** The region R is simply connected.

True **False** The principal complex logarithm $\text{Log}(z)$ is continuous on R.

2. (12 points) Solve the following problems related to holomorphic functions:

(a) For $f(z) = -iz + z^2\overline{z}^2$, show that the complex derivative f' does not exist at any point except $z = 0$, AND show $f'(0) = -i$.

(b) For a function $f(z) = f(x+iy)$ with x, y real, let $f^*(z) = \overline{f(\overline{z})} = \overline{f(x-iy)}$: for example, if $f(z) = 3i + (2+i)z^2$ then $f^*(z) = -3i + (2-i)z^2$. Show that if f is holomorphic on the open region R, then f^* is holomorphic on the conjugate region $\overline{R} = \{\overline{a} : a \in R\}$.

3. (12 points) Consider the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{n(n+1)}{2} z^n = z + 3z^2 + 6z^3 + 10z^4 + 15z^5 + 21z^6 + \cdots .$$

(a) Find the radius and disc of convergence for $f(z)$.

(b) Find a power series for a function $F(z)$ with $F'(z) = f(z)$ and $F(0) = 2025$.

(c) Find the terms in a Laurent series expansion centered at $z = 0$ for $1/f(z)$ up to order 1 (i.e., up through the z^1-term).

(d) Suppose γ is a closed rectifiable contour lying inside the disc of convergence for $f(z)$. What is the value of $\int_\gamma f(z)\, dz$? Briefly explain.

4. (12 points) Compute each of the following complex line integrals on the given non-closed contours γ:

(a) $\displaystyle\int_\gamma |z|^2\,dz$ where $\gamma(t) = 3e^{it}$ for $0 \le t \le \pi$.

(b) $\displaystyle\int_\gamma [\mathrm{Re}(z)]^2\,dz$ where γ is the line segment from i to $3 - 4i$.

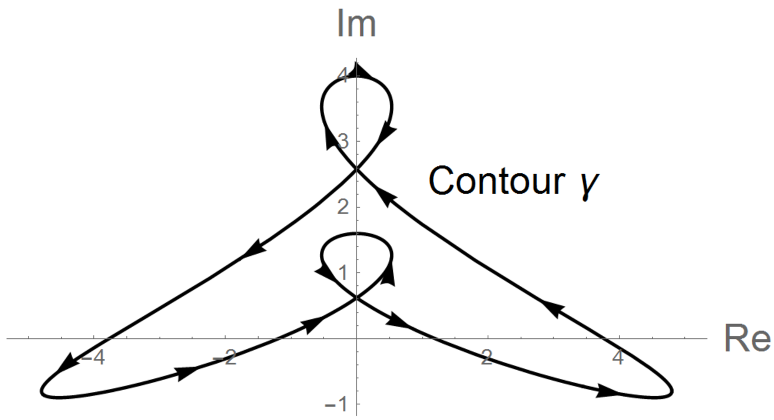
(c) $\displaystyle\int_\gamma 3z^2\,dz$ where $\gamma(t) = te^t + i\sin^{2025}(2\pi t)$ for $0 \le t \le 1$.

5. (8 points) Let $f(z) = \dfrac{e^{2z}}{z^4}$.

(a) Find a Laurent series expansion for $f(z)$ centered at $z = 0$ and give explicitly the terms in the expansion up to order 0 (i.e., up to the constant term).

(b) Evaluate the contour integral $\displaystyle\int_\gamma f(z)\,dz$ where γ is the counterclockwise boundary of the square with vertices $\pm 2 \pm 2i$.

6. (12 points) Consider the contour γ plotted below.



(a) Find the winding numbers of γ around 0, around i, and around $3i$.

(b) Find $\displaystyle\int_\gamma \frac{1}{z}\,dz$.

(c) Find $\displaystyle\int_\gamma \frac{e^z}{z-i}\,dz$.

(d) Find $\displaystyle\int_\gamma \frac{1}{z^2+1}\,dz$.

7. (10 points) Prove the following things:

(a) Let $f(z) = \sum_{n=0}^{\infty} \frac{z^{3n}}{(3n)!} = 1 + \frac{z^3}{3!} + \frac{z^6}{6!} + \cdots$. Show that $f(z)$ is holomorphic on the entire complex plane and that it satisfies the differential equation $f''(z) + f'(z) + f(z) = e^z$.

(b) Suppose that $f(z)$ is holomorphic on a simply-connected region R with counterclockwise boundary γ. If z_0 is any interior point of R, prove that $\int_\gamma \frac{f'(z)}{z - z_0}\,dz = \int_\gamma \frac{f(z)}{(z - z_0)^2}\,dz$. [Hint: Use the differentiation-via-integration formula.]

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