1. (a) Zeroes occur when $e^{1/(z-1)} = 1$ so that $1/(z-1) = 2\pi ik$ so that $z = 1 + 1/(2\pi ik)$ for nonzero integers k. Poles occur when the denominator is zero and the numerator is not, which occurs when $z = \boxed{0,\ -4}$. An essential singularity occurs when the exponent of e is unbounded, which occurs at $z = \boxed{1}$.

   (b) Both poles are simple so the residue at $z = 0$ is $\lim_{z\to 0} zf(z) = \lim_{z\to 0} \dfrac{e^{1/(z-1)} - 1}{z+4} = \boxed{\dfrac{e^{-1} - 1}{4}}$ and the residue at $z = -4$ is $\lim_{z\to -4}(z+4)f(z) = \lim_{z\to -4} \dfrac{e^{1/(z-1)} - 1}{z} = \boxed{\dfrac{e^{-1/5} - 1}{-4}}$.

   (c) The radius is the smallest distance to a point where $f(z)$ is not holomorphic, which are the poles $z = 0$ and $z = -4$, and the essential singularity $z = 1$. The minimal distance is $\boxed{1}$ to the pole at $z = 0$.

   (d) By the residue theorem, this is $2\pi i$ times the sum of the residues inside the circle. Since the only singularity inside the circle is at $z = 0$, the integral is $\boxed{2\pi i \cdot (e^{-1} - 1)/4}$.

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2. (a) We have $e^z + e^{1/z} = \boxed{\cdots + \frac{1}{24}z^{-4} + \frac{1}{6}z^{-3} + \frac{1}{2}z^{-2} + z^{-1} + 2 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \cdots}$.

   (b) With $w = z+1$, $f(z) = -\dfrac{1}{w} \cdot \dfrac{1}{1-w} = -\dfrac{1}{w} \sum_{n=0}^{\infty} w^n = \boxed{\sum_{n=-1}^{\infty} -(z+1)^n}$.

   (c) With $w = z+1$, $f(z) = \dfrac{1}{w^2} \cdot \dfrac{1}{1-1/w} = \dfrac{1}{w^2} \sum_{n=0}^{\infty} w^{-n} = \boxed{\sum_{n=-\infty}^{-2} (z+1)^n}$.

   (d) There is a pole of order 2 at $z = 0$ and a pole of order 1 at $z = -3$. The residue at $z = -3$ is $\lim_{z\to -3}(z+3)f(z) = \lim_{z\to -3} \dfrac{z^3+1}{z^2} = \boxed{-26/9}$. The residue at $z = 0$ is $\dfrac{1}{1!} \lim_{z\to 0} \dfrac{d}{dz}[z^2 f(z)] = \lim_{z\to 0} \dfrac{3z^2(z+3) - (z^3+1)}{(z+3)^2} = \boxed{-1/9}$.

   (e) There are simple poles at each zero of $\sin 2z$, which occur for $z = \boxed{k\pi/2}$ for integers k. At $z = k\pi/2$, the residue is $\dfrac{e^z}{2\cos 2z} = \dfrac{e^{k\pi/2}}{2\cos(k\pi)} = \boxed{(-1)^k e^{k\pi/2}/2}$ using the simple pole residue formula (the residue of $p(z)/q(z)$ is $p(z_0)/q'(z_0)$).

   (f) The function has simple poles at $z = \pm 45i$ neither of which is inside the contour, so the integral is $\boxed{0}$ by the residue theorem.

   (g) Only the pole at $z = 45i$ is inside the contour. The residue there is $\lim_{z\to 45i}(z-45i)f(z) = \dfrac{1}{90i}$ so the integral is $2\pi i/(90i) = \boxed{\pi/45}$ by the residue theorem.

   (h) Only the pole at $z = 5$ is inside the contour. The residue there is $\dfrac{1}{3!} \lim_{z\to 5} \dfrac{d^3}{dz^3}[(z-5)^4 f(z)] = \dfrac{1}{6} \lim_{z\to 5} \dfrac{d^3}{dz^3}[z^{-3}] = \dfrac{(-3)(-4)(-5)}{3! \cdot 5^6} = -\dfrac{2}{5^5}$ so the integral is $2\pi i \cdot (-2/5^5) = \boxed{-4\pi i/3125}$ by the residue theorem.

   (i) The denominator has simple zeroes at $z = k\pi$, but since the numerator is zero at $z = 0$, there is not a pole there. So the only poles inside the contour are at $z = \pm\pi$. The residue at $z = k\pi$ is $k\pi/\cos(k\pi)$ by the simple pole formula, so the integral is $2\pi i \cdot (\pi - \pi) = \boxed{0}$.

   (j) Substituting $z = e^{i\theta}$ yields $\int_0^{2\pi} r(\cos\theta, \sin\theta)\, d\theta = \int_\gamma f(z)\, dz$ where γ is the unit circle and $f(z) = r(\dfrac{z+z^{-1}}{2}, \dfrac{z-z^{-1}}{2i}) \cdot \dfrac{1}{iz} = \dfrac{1}{i} \dfrac{1}{2z^2+5z+2}$ which has simple poles at $z = -1/2, -2$. Only $z = -1/2$ is inside the unit circle, and the residue of f there is $\lim_{z\to -1/2} \dfrac{1}{i} \dfrac{1}{4z+5} = \dfrac{1}{3i}$. Then by the residue theorem $I = \int_\gamma f(z)\, dz = 2\pi i \cdot (\dfrac{1}{3i}) = \boxed{2\pi/3}$.

(k) Integrate along the semicircular contour of radius R, with $\gamma_1 = [-R, R]$ and γ_2 the upper semicircle from R to $-R$. Take $f(z) = 1/(z^4 + 4)$. Since f has simple poles at the solutions of $z^4 = 4$ which are $z = \pm 1 \pm i$ we see that $\pm 1 + i$ are inside γ. The residue at $1 + i$ is $\lim_{z \to 1+i} 1/(4z^3) = (-1 - i)/16$ and the residue at $-1 + i$ is $\lim_{z \to -1+i} 1/(4z^3) = (1 - i)/16$ so $\int_\gamma f(z)\,dz = 2\pi i \cdot [(-1-i)/16 + (1-i)/16] = \pi/4$. Since $I = \lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz$ and $\left|\int_{\gamma_2} f(z)\,dz\right| \le 2\pi \cdot \frac{1}{R^4-4} R = O(R^{-3})$ taking $R \to \infty$ yields $I = \boxed{\pi/4}$.

(l) Integrate along the semicircular contour of radius R, with $\gamma_1 = [-R, R]$ and γ_2 the upper semicircle from R to $-R$. Take $f(z) = e^{iz}/(z^2 + 4)$. Since f has simple poles at $z = \pm 3i$ we see $z = 3i$ is inside γ and the residue there is $\lim_{z\to 3i} e^{iz}/(z + 3i) = e^{-3}/(6i)$ so $\int_\gamma f(z)\,dz = 2\pi i \cdot [e^{-3}/(6i)] = \pi e^{-3}/3$. Since $I = \text{Re}[\lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz]$ and $\left|\int_{\gamma_2} f(z)\,dz\right| \le 2\pi \cdot \frac{1}{R^2-9} R = O(R^{-1})$ taking $R \to \infty$ and then taking the real part yields $I = \boxed{\pi e^{-3}/3}$.

(m) By the argument principle / zero-and-pole counting, $\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i[Z - P]$ where Z is the number of zeroes and P is the number of poles inside γ. Here, $f(z)$ has a zero of order 3 at $z = 0$, zeroes of order 1 at $z = \pm i$, and a pole of order 5 at $z = 3$. Only the zeroes are inside the circle, so the integral equals $2\pi i \cdot 4 = \boxed{8\pi i}$.

(n) By the argument principle / zero-and-pole counting, $\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i[Z - P]$ where Z is the number of zeroes and P is the number of poles inside γ. Here, $f(z)$ has a zero of order 3 at $z = 0$, zeroes of order 1 at $z = \pm i$, and a pole of order 5 at $z = 3$. The poles are also inside the circle now, so the integral equals $2\pi i \cdot (4 - 5) = \boxed{-2\pi i}$.

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3. Recall that the Cauchy estimate says that if $|f(z)| \le M$ on $|z - z_0| = r$, then $\left|f^{(n)}(z_0)\right| \le M \cdot n!/r^n$. Applying it here yields $\left|f^{(n)}(0)\right| \le r^{3/2-n} \cdot n!$, which for $n \ge 2$ tends to zero as $r \to \infty$. Thus, we see $f^{(n)}(0) = 0$ for $n \ge 2$, and so we have the power series $f(z) = a + bz$. But the given condition also requires $f(0) = 0$ so in fact $a = 0$ hence $f(z) = bz$ as claimed.
Alternatively, the condition requires $f(0) = 0$ and that $|f(z)/z| \le 1$ for all $z \ne 0$, so $f(z)/z$ has a removable singularity at $z = 0$ since it is bounded nearby. Removing it, we see $g(z) = f(z)/z$ is entire and has $|g(z)| \le 1$ so $g(z)$ is a bounded entire function hence is constant by Liouville's theorem.

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4. Since $q(z)$ is nonzero, it has finitely many zeroes and so $f(z)$ has finitely many poles. If γ is the counterclockwise circle $|z| = R$ where R is large enough that γ encloses all of the poles, then by the residue theorem we know that $\frac{1}{2\pi i}\int_\gamma f(z)\,dz$ is the sum of the residues of $f(z)$. But for such R, using the arclength estimate we see that $\left|\int_\gamma f(z)\,dz\right| = O(2\pi R \cdot R^{\deg p}/R^{\deg q}) = O(R^{1+\deg p - \deg q})$. By hypothesis the power is negative, so as $R \to \infty$ the integral goes to zero; since the integral is constant for large R, the integral and hence the sum of residues must equal zero.

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5. Since $f(z) = z^2 + 3z - 4$ is holomorphic, by the maximum modulus principle the maximum occurs on the boundary circle $|z| = 2$. For $z = 2e^{i\theta}$ we see $\left|z^2 + 3z - 4\right|^2 = (4e^{2i\theta} + 6e^{i\theta} - 4)(4e^{-2i\theta} + 6e^{-i\theta} - 4) = 68 - 16(e^{2i\theta} + e^{-2i\theta}) = 68 - 32\cos 2\theta$. The maximum value clearly occurs for $\cos 2\theta = -1$, and then the maximum of $\left|z^2 + 3z - 4\right|$ is $\sqrt{100} = 10$.

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6. Per the hint observe that $\left|e^{f(z)}\right| = e^{\text{Re}[f(z)]} \le e^{2025}$ and thus $e^{f(z)}$ is a bounded entire function hence it is constant by Liouville's theorem. If $e^{f(z)} = C$ then the image of f lies in the set of possible logarithms $\log(C)$, but no such set can be an open set. So by the open mapping theorem, f must be constant.

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7. (a) Suppose M is not strictly increasing, so that $M(a) \ge M(b)$ for some $a < b$. This means there is some point z_a with $|z_a| = a$ such that $|f(z)| \le |f(z_a)|$ for all z on the circle $|z| = b$. However, this contradicts the maximum modulus principle applied to the region $|z| \le b$, which says that if f is nonconstant then the maximum modulus can only occur on the boundary of the disc.

(b) If $M(r)$ does not tend to ∞ then it must be bounded, since it is increasing by part (a). But then f would be a bounded entire function hence constant by Liouville's theorem, contradiction.
