1. Let $f(z) = \dfrac{e^{1/(z-1)} - 1}{z^2 + 4z}$.
   (a) Identify all zeroes, poles, and essential singularities of f.
   (b) Find the residue of f at each pole.
   (c) Find the radius of convergence of the power series for $f(z)$ centered at $z = i$.
   (d) Find $\int_\gamma f(z)\,dz$ where γ is the counterclockwise circle $|z| = 10^{-100}$.

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2. Calculate, with justification:
   (a) The terms in the Laurent series for $f(z) = z/(z-2)$ on the region $0 < |z-1| < 1$ from order -2 to 2.
   (b) The Laurent series for $f(z) = 1/(z + z^2)$ on the region $0 < |z+1| < 1$.
   (c) The Laurent series for $f(z) = 1/(z + z^2)$ on the region $|z+1| > 1$.
   (d) All poles for $f(z) = (z^3 + 1)/(z^3 + 3z^2)$, their orders, and the residue at each.
   (e) All poles for $f(z) = e^z/(\sin 2z)$, their orders, and the residue at each.
   (f) $\int_\gamma \dfrac{1}{z^2 + 2025}\,dz$ where γ is the counterclockwise circle $|z| = 1$.
   (g) $\int_\gamma \dfrac{1}{z^2 + 2025}\,dz$ where γ is the counterclockwise circle $|z - 25i| = 25$.
   (h) $\int_\gamma \dfrac{1}{z^3(z-5)^4}\,dz$ where γ is the counterclockwise circle $|z - 4| = 3$.
   (i) $\int_\gamma \dfrac{z}{\sin z}\,dz$ where γ is the counterclockwise circle $|z| = 4$.
   (j) $\int_0^{2\pi} \dfrac{1}{5 + 4\cos\theta}\,d\theta$.
   (k) $\int_{-\infty}^{\infty} \dfrac{1}{z^4 + 4}\,dz$.
   (l) $\int_{-\infty}^{\infty} \dfrac{\cos z}{z^2 + 9}\,dz$.
   (m) $\int_\gamma \dfrac{f'(z)}{f(z)}\,dz$ where $f(z) = \dfrac{z^3(z^2+1)e^{3z}}{(z-3)^5}$ and γ is the counterclockwise circle $|z| = 2$.
   (n) $\int_\gamma \dfrac{f'(z)}{f(z)}\,dz$ where $f(z) = \dfrac{z^3(z^2+1)e^{3z}}{(z-3)^5}$ and γ is the counterclockwise circle $|z| = 4$.

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3. Suppose that $f(z)$ is entire and $|f(z)| \le |z|$ for all z. Show that $f(z) = bz$ for some constant b. [Hint: Cauchy estimates.]

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4. Suppose that $p(z)$ and $q(z)$ are polynomials with $1 + \deg p < \deg q$. Show that the sum of the residues of $f(z) = p(z)/q(z)$, over all poles, is equal to zero. [Hint: Integrate around $|z| = R$ as $R \to \infty$.]

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5. Prove that the maximum value of $\left|z^2 + 3z - 4\right|$ on the region $|z| \le 2$ is 10.

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6. Suppose that $f(z)$ is entire and $\mathrm{Re}[f(z)] \le 2025$ for all $z \in \mathbb{C}$. Show $f(z)$ is constant. [Hint: Consider $\left|e^{f(z)}\right|$.]

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7. Suppose that $f(z)$ is a nonconstant entire function. For real $r \ge 0$, let $M(r)$ be the maximum value of $|f(z)|$ on the circle $|z| = r$.
   (a) Show that $M(r)$ is a strictly increasing function: namely, that $M(a) < M(b)$ for any $a < b$.
   (b) Show that $M(r) \to \infty$ as $r \to \infty$.
