1. For each function $f(z)$, identify (i) all zeroes and their orders, (ii) all removable singularities, (iii) all poles and their orders, and (iv) all essential singularities:

   (a) $f(z) = e^z - 1$.

   - We see that f is entire so it has no singularities or poles. The zeroes of f are at $z = \boxed{2\pi i k}$ for each $k \in \mathbb{Z}$, and since $f'(2\pi i k) = 1$ all of the zeroes have order $\boxed{1}$.

   (b) $f(z) = \dfrac{1}{e^z - 1}$.

   - Taking the reciprocal swaps zeroes and poles, so now f has poles at $z = \boxed{2\pi i k}$ for each $k \in \mathbb{Z}$ each of which has order $\boxed{1}$. There are no zeroes and no other singularities.

   (c) $f(z) = \dfrac{e^z - 2}{e^z - 1}$.

   - From (b) and writing $f(z) = 1 - \dfrac{1}{e^z - 1}$ we see that f still has poles of order $\boxed{1}$ at $z = \boxed{2\pi i k}$ for each $k \in \mathbb{Z}$ and no other singularities.
   - Also, f has zeroes at $z = \boxed{\ln 2 + 2\pi i k}$ for $k \in \mathbb{Z}$ and since $f'(\ln 2 + 2\pi i k) \neq 0$ they all have order $\boxed{1}$.

   (d) $f(z) = \dfrac{z^5 - z}{z^3 - 2z^2 + z}$.

   - Since $f(z)$ is rational it has no essential singularities, and to find zero and pole orders we can just count multiplicities of roots in the numerator and denominator. The numerator has simple zeroes at $z = 0, \pm 1, \pm i$ while the denominator has a simple zero at $z = 0$ and a double zero at $z = 1$.
   - So we see that overall, f has a removable singularity at $z = \boxed{0}$, zeroes of order 1 at $z = \boxed{i, -i, -1}$ and a pole of order 1 at $z = \boxed{1}$.

   (e) $f(z) = \dfrac{\sin(z^2)}{z^3}$.

   - Since $\sin(z^2) = 0$ for $z = \pm\sqrt{2\pi k}, \pm i\sqrt{2\pi k}$ for nonnegative integers k, and $z^3 = 0$ for $z = 0$ we see that the only singularity is at $z = 0$.
   - Furthermore since $f'(z) = 2z^{-2}\cos(z^2) - 3z^{-4}\sin(z^2)$ we see that $f'(\pm\sqrt{2k\pi}) \neq 0$ for $k \neq 0$. So all of the zeroes at $\boxed{\pm\sqrt{2\pi k}, \pm i\sqrt{2\pi k}}$ for $k > 0$ have order $\boxed{1}$.
   - At $z = 0$ f has Laurent expansion $z^{-3}(z^2 - \frac{1}{2}z^4 + \cdots) = z^{-1} - \frac{1}{2}z + \cdots$ so f has a pole of order $\boxed{1}$.

   (f) $f(z) = (z^2 - 1)\sin(\pi/z)$.

   - The only singularity is at $z = 0$ and it is an essential singularity since $\sin(\pi/z)$ is bounded but oscillatory on the real axis as $z \to 0$, and this means the singularity is neither removable nor a pole. (Alternatively we could use the Laurent expansion.)
   - The zeroes of $z^2 - 1$ are order 1 and occur for $z = \pm 1$. The zeroes for $\sin(\pi/z)$ are order 1 and occur for $z = 1/k$ for integers k.
   - So, the zeroes of $f(z)$ at $z = \pm 1$ have order $\boxed{2}$ and the zeroes at $z = 1/k$ for $k \neq \pm 1$ have order $\boxed{1}$.

   (g) $f(z) = \dfrac{e^{\pi z}\sin(\pi z)}{z(z^2 - 1)^3}$.

   - The singularities are at $z = \pm 1$. Note $e^{\pi z}$ has no zeroes or poles and $\sin(\pi z)$ has simple zeroes at $z = k$ for integers k, while $z(z^2 - 1)^2$ has a zero of order 1 at $z = 0$ and zeroes of order 3 at $z = \pm 1$.
   - At $z = 0$ the simple zeroes of the numerator and denominator cancel yielding a removable singularity.
   - At $z = \pm 1$ the simple zero of the numerator cancels one of the triple zeroes from the denominator yielding poles of order $\boxed{2}$, and at $z = k$ for $k \neq 0, \pm 1$ there is just a zero of order $\boxed{1}$.

2. Calculate the residue of each function $f(z)$ at the given point:

(a) $f(z) = e^z/\sin(z)$ at $z = 0$.

- Since e^z is nonzero and $\sin z$ has a simple zero at $z = 0$, by the simple pole formula with $g(z) = e^z$ and $h(z) = \sin z$, the residue is $g(\pi)/h'(\pi) = \boxed{1}$.

(b) $f(z) = e^z/\sin(z)$ at $z = \pi$.

- Since e^z is nonzero and $\sin z$ has a simple zero at $z = \pi$, by the simple pole formula with $g(z) = e^z$ and $h(z) = \sin z$, the residue is $g(\pi)/h'(\pi) = \boxed{-e^\pi}$.

(c) $f(z) = e^z/(z+1)^2$ at $z = -1$.

- Since $f(z) = \dfrac{1}{e}\left[(z+1)^{-2} + (z+1)^{-1} + \dfrac{1}{2} + \dfrac{1}{6}(z+1) + \cdots\right]$ the residue is $\boxed{1/e}$.
- Or, as f has a pole of order 2 we compute $\lim_{z\to-1} \dfrac{d}{dz}[(z+1)^2 f(z)] = \lim_{z\to-1} e^z = \boxed{1/e}$.

(d) $f(z) = e^z/\sin(z^2)$ at $z = 0$.

- Since $f(z) = z^{-2} + z^{-1} + \dfrac{1}{2} + \dfrac{1}{6}z + \cdots$ the residue is $\boxed{1}$.

(e) $f(z) = \dfrac{1}{z^3(z+1)^4}$ at $z = 0$.

- Since f has a pole of order 3 the residue is $\dfrac{1}{2!}\lim_{z\to0} \dfrac{d^2}{dz^2}[z^3 f(z)] = \dfrac{1}{2}\lim_{z\to0} \dfrac{20}{(z+1)^6} = \boxed{10}$.

(f) $f(z) = \dfrac{1}{z^3(z+1)^4}$ at $z = -1$.

- Since f has a pole of order 4 the residue is $\dfrac{1}{3!}\lim_{z\to-1} \dfrac{d^3}{dz^3}[(z+1)^4 f(z)] = \dfrac{1}{6}\lim_{z\to-1} \dfrac{-60}{z^6} = \boxed{-10}$.

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3. Consider the function $f(z) = \dfrac{e^{1/z}}{1-z^2}$ and observe that $f(z)$ has an essential singularity at $z = 0$.

(a) Identify the poles for $f(z)$ and compute the order and residue at each pole.

- We see that f is undefined at $z = 0$ and $z = \pm1$, and since $1/f(z) = \dfrac{1-z^2}{e^{1/z}}$ is holomorphic at $z = \pm1$ but not at $z = 0$, that means f has poles at $z = \boxed{\pm1}$.
- As $\lim_{z\to-1}(z+1)f(z) = \lim_{z\to-1} \dfrac{e^{1/z}}{-(z-1)} = \dfrac{e}{2}$ the pole $z = -1$ has order $\boxed{1}$ with residue $\boxed{e/2}$.
- As $\lim_{z\to1}(z-1)f(z) = \lim_{z\to1} \dfrac{e^{1/z}}{-(z+1)} = -\dfrac{e}{2}$ the pole $z = 1$ has order $\boxed{1}$ with residue $\boxed{-e/2}$.

(b) Find the terms from order -4 to 0 of the Laurent expansion for $f(z)$ on the region $|z| > 1$.

- We have $f(z) = \dfrac{1}{z^2} \cdot \dfrac{e^{1/z}}{1-1/z^2} = z^{-1}(1 + z^{-1} + \dfrac{1}{2!}z^{-2} + \dfrac{1}{3!}z^{-3} + \cdots)(1 + z^{-2} + z^{-4} + \cdots)$ $= \boxed{\cdots + \dfrac{3}{2}z^{-4} + z^{-3} + z^{-2}}$ after multiplying out.

(c) Find the terms from order -2 to 2 of the Laurent expansion for $f(z)$ on the region $0 < |z| < 1$. [The coefficients will be infinite sums. You can express them in terms of the functions sinh and cosh; please do so.]

- For $|z| < 1$ we have $e^{1/z} = 1 + z^{-1} + \dfrac{1}{2!}z^{-2} + \dfrac{1}{3!}z^{-3} + \cdots$ and $\dfrac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + z^8 + \cdots$.
- So expanding out and then collecting terms yields $e^{1/z}(z^2-1)$

$$= (1+z^{-1}+\frac{1}{2!}z^{-2}+\frac{1}{3!}z^{-3}+\cdots)+z^2(1+z^{-1}+\frac{1}{2!}z^{-2}+\frac{1}{3!}z^{-3}+\cdots)+z^4(1+z^{-1}+\frac{1}{2!}z^{-2}+\frac{1}{3!}z^{-3}+\cdots)+\cdots$$

$$= (1+z^{-1}+\frac{1}{2!}z^{-2}+\frac{1}{3!}z^{-3}+\cdots)+(z^2+z^1+\frac{1}{2!}+\frac{1}{3!}z^{-1}+\cdots)+(z^4+z^3+\frac{1}{2!}z^2+\frac{1}{3!}z^1+\frac{1}{4!}+\frac{1}{5!}z^{-1}+\cdots)+\cdots$$

$$= \cdots+(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots)z^{-2}+(1+\frac{1}{3!}+\frac{1}{5!}+\cdots)z^{-1}+(1+\frac{1}{2!}+\frac{1}{4!}+\cdots)+(1+\frac{1}{3!}+\frac{1}{5!}+\cdots)z+\cdots.$$

- In general, $a_{\pm n} = \frac{1}{n!}+\frac{1}{(n+2)!}+\frac{1}{(n+4)!}+\cdots = \left\{\begin{matrix}\cosh 1, & n \text{ even} \\ \sinh 1, & n \text{ odd}\end{matrix}\right\} - \frac{1}{0!}-\frac{1}{2!}-\cdots-\frac{1}{(n-2)!}$.
- So the desired terms are $\boxed{\cdots+(\cosh 1-1)z^{-2}+(\sinh 1)z^{-1}+\cosh 1+(\sinh 1)z+(\cosh 1-1)z^2+\cdots}$.

(d) Find $\int_\gamma f(z)\,dz$ where γ is the counterclockwise circle $|z| = 0.1$.

- Since f has a convergent Laurent expansion on the region $0 < |z| < 1$, by our results on Laurent expansions we see that $\int_\gamma f(z)\,dz = 2\pi i \cdot W_\gamma(0)\cdot a_{-1} = \boxed{2\pi i \sinh(1)}$ where a_{-1} is the coefficient of z^{-1} as computed in part (c).

(e) Find $\int_\gamma f(z)\,dz$ where γ is the counterclockwise circle $|z-1| = 0.1$.

- By the residue theorem, since f is meromorphic inside γ and has a single pole at $z = 1$ there, we have $\int_\gamma f(z)\,dz = 2\pi i\cdot W_\gamma(1)\cdot \mathrm{res}_f(1) = \boxed{-\pi i e}$.

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4. Calculate the following contour integrals:

(a) $\int_\gamma \frac{1}{z^2-2025z}\,dz$ where γ is the counterclockwise circle $|z| = 1$.

- The integrand is meromorphic with simple poles at $z = 0$ and $z = 2025$. Only the pole at $z = 0$ is contained in the circle, so by the residue theorem we see $\int_\gamma f(z)\,dz = 2\pi i\cdot \mathrm{Res}_f(0)$.
- The residue at $z = 0$ is $\lim_{z\to 0} z\cdot\frac{1}{z^2-2025z} = \lim_{z\to 0}\frac{1}{z-2025} = -\frac{1}{2025}$, so the integral is $\boxed{-\frac{2\pi i}{2025}}$.

(b) $\int_\gamma \frac{1}{z^2-2025z}\,dz$ where γ is the counterclockwise circle $|z| = 5000$.

- The integrand is meromorphic with simple poles at $z = 0$ and $z = 2025$. Now both poles are contained in the circle, so by the residue theorem we see $\int_\gamma f(z)\,dz = 2\pi i\cdot[\mathrm{Res}_f(0)+\mathrm{Res}_f(2025)]$.
- The residue at $z = 2025$ is $\lim_{z\to 2025}\frac{z-2025}{z^2-2025z} = \lim_{z\to 2025}\frac{1}{z} = \frac{1}{2025}$, so the integral is $\boxed{0}$.

(c) $\int_\gamma \frac{e^z}{z(z-1)(z-2)}\,dz$ where γ is the counterclockwise circle $|z| = 3$.

- The integrand is meromorphic with simple poles at $z = 0$, $z = 1$, and $z = 2$, each of which are in the circle, so by the residue theorem we see $\int_\gamma f(z)\,dz = 2\pi i\cdot[\mathrm{Res}_f(0)+\mathrm{Res}_f(1)+\mathrm{Res}_f(2)]$.
- The residues at $z = 0, 1, 2$ are $\lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{e^z}{(z-1)(z-2)} = \frac{1}{2}$, $\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{e^z}{z(z-2)} = -e$, and $\lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{e^z}{z(z-1)} = \frac{e^2}{2}$ respectively.
- Thus, the value of the integral is $\boxed{\pi i(1-2e+e^2)}$.

(d) $\int_\gamma \frac{1}{e^z-1}\,dz$ where γ is the counterclockwise boundary of the square with vertices $(\pm\frac{5}{2}\pm\frac{5}{2}i)\pi$.

- The integrand $f(z)$ is meromorphic with poles at $z = 2\pi i k$ for each integer k.
- The poles with $k = 0, \pm 1, \pm 2$ lie inside the square while the others lie outside, so by the residue theorem we see $\int_\gamma f(z)\,dz = 2\pi i\cdot[\mathrm{Res}_f(0)+\mathrm{Res}_f(2\pi i)+\mathrm{Res}_f(4\pi i)+\mathrm{Res}_f(-2\pi i)+\mathrm{Res}_f(-4\pi i)]$.
- For each residue, we compute $(z-2\pi i k)f(z) = \frac{z-2\pi i k}{e^z-1}$ and then take the limit as $z\to 2\pi i k$ to obtain $\mathrm{Res}_f(2\pi i k) = \lim_{z\to 2\pi i k}\frac{z-2\pi i k}{e^z-1} = \lim_{z\to 2\pi i k}\frac{1}{e^z} = 1$ by L'Hôpital's rule.

- Since each residue is 1, the integral is $\boxed{10\pi i}$.

(e) $\int_\gamma \dfrac{\sin \pi z}{\sin z}\, dz$ where γ is the counterclockwise circle $|z-1| = 4$.

- Since the denominator function $\sin z$ has simple zeroes at $z = k\pi$ for integers k, we see that the function has singularities at $z = k\pi$ for integers k. But since the numerator $\sin \pi z$ has a simple zero at $z = 0$, in fact the singularity at $z = 0$ is removable.
- Inside the circle $|z| = 4$, therefore, the function has only a single pole, at $z = \pi$, so by the residue theorem, we see $\int_\gamma f(z)\, dz = 2\pi i \cdot \text{Res}_f(\pi)$.
- The residue at $z = \pi$ is $\lim_{z\to\pi}(z-\pi)\dfrac{\sin \pi z}{\sin z} = \sin(\pi^2)\lim_{z\to\pi}\dfrac{z-\pi}{\sin z} = \sin(\pi^2)\lim_{z\to\pi}\dfrac{1}{\cos z} = -\sin(\pi^2)$ by L'Hôpital's rule. Thus, the value of the integral is $\boxed{-2\pi i \sin(\pi^2)}$.

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5. Suppose that $f(z)$ is bounded and holomorphic on $\mathbb{C}\backslash\{0\}$. Show that $f(z)$ is constant.

- By our results on removable singularities, since $f(z)$ is bounded and holomorphic on a punctured disc containing 0, it must have a removable singularity there.
- Hence if we define $g(z)$ to be the function obtained by removing the singularity, then $g(z) = f(z)$ on $\mathbb{C}\backslash\{0\}$ and g is holomorphic at 0.
- In other words, $g(z)$ is a bounded holomorphic function on the entire complex plane. Hence by Liouville's theorem, it is constant, and so f is also constant since it takes the same values as g.

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6. Suppose $f(z)$ is meromorphic on a region R and that f has an isolated singularity at $z = z_0$.

(a) If $\lim_{z\to z_0}(z-z_0)f(z) = 0$, show that f has a removable singularity at z_0.

- By considering the Laurent expansion near z_0, we see that $\lim_{z\to z_0}(z-z_0)f(z)$ can only exist if $(z-z_0)f(z)$ has no terms of negative degree in $z - z_0$: thus $(z-z_0)f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$.
- Furthermore, in that case we have $\lim_{z\to z_0}(z-z_0)f(z) = a_0 = 0$.
- Thus $(z-z_0)f(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n$ and so $f(z) = \sum_{n=0}^{\infty} a_{n+1}(z-z_0)^n$ has no terms of negative order in its Laurent expansion, meaning it has a removable singularity at z_0.

(b) If f has a zero of order k at z_0, show $f'(z)/f(z)$ has a simple pole at z_0 with residue k.

- If $f(z) = a_k(z-z_0)^k + \cdots$ then $f'(z) = ka_k(z-z_0)^{k-1} + \cdots$.
- Then the Laurent series for $f(z)^{-1} = \dfrac{1}{a_k}(z-z_0)^{-k} + \cdots$, and so $\dfrac{f'(z)}{f(z)} = k(z-z_0)^{-1} + \cdots$.
- This means $f'(z)/f(z)$ has a simple pole at z_0, and its residue there is the coefficient of $(z-z_0)^{-1}$ which is k.

(c) If f has a pole of order k at z_0, show $f'(z)/f(z)$ has a simple pole at z_0 with residue $-k$.

- The same calculation in (a) works: if $f(z) = a_k(z-z_0)^{-k} + \cdots$ then $f'(z) = -ka_k(z-z_0)^{-k-1} + \cdots$.
- Then the Laurent series for $f(z)^{-1} = -\dfrac{1}{a_k}(z-z_0)^k + \cdots$, and so $\dfrac{f'(z)}{f(z)} = -k(z-z_0)^{-1} + \cdots$.
- So $f'(z)/f(z)$ has a simple pole at z_0, and its residue there is $-k$.

(d) If f has a pole at z_0, show that $\text{Re}(f)$ and $\text{Im}(f)$ take arbitrarily large positive and negative values as $z \to z_0$. [Hint: If $f(z) = re^{i\theta}(z-z_0)^{-k} + \cdots$, take $z = z_0 + t^{1/k}e^{i\theta/k}$ as $t \to 0$ in different directions.]

- Suppose the pole has order k, so $f(z) = re^{i\theta}(z-z_0)^{-k} + \sum_{n=1-k}^{\infty} a_n(z-z_0)^n$.
- Then $\lim_{z\to z_0}\dfrac{f(z)}{re^{i\theta}(z-z_0)^{-k}} = \lim_{z\to z_0}[1+\sum_{n=1}^{\infty} a_{n-k}(z-z_0)^n] = 1$ meaning that $\dfrac{f(z)}{re^{i\theta}(z-z_0)^{-k}} = 1 + \epsilon$ with $\epsilon \to 0$ as $z \to z_0$.
- So for $a_k = re^{i\theta}$ where $r > 0$, if we take $z = z_0 + t^{1/k}e^{i\theta/k}$ as $t \to 0$, then $f(z) = rt^{-1}(1+\epsilon)$.
- Hence for $t \to 0$ along the positive real axis we see $\text{Re}[f(z)] \to +\infty$, for $t \to 0$ along the negative real axis we see $\text{Re}[f(z)] \to -\infty$, for $t \to 0$ along the negative imaginary axis we see $\text{Im}[f(z)] \to +\infty$, and for $t \to 0$ along the positive imaginary axis we see $\text{Im}[f(z)] \to -\infty$.

- This means $\text{Re}(f)$ and $\text{Im}(f)$ take arbitrarily large positive and negative values as $z \to z_0$, as required.

(e) If f has an essential singularity at z_0, show that e^f also has an essential singularity at z_0.

- By Casorati-Weierstrass, since f has an essential singularity at z_0, there exists a sequence $\{z_n\}$ with $z_n \to z_0$ and $f(z_n) \to 0$ as $n \to \infty$. Then $e^{f(z_n)} \to 1$ as $n \to \infty$.
- Likewise, there also exists a sequence $\{z_n\}$ with $z_n \to z_0$ and $f(z_n) \to +\infty$ (positive real infinity) as $n \to \infty$. Then $e^{f(z_n)} \to +\infty$ as well.
- So e^f takes values that are both bounded and unbounded near 0, so e^f must have an essential singularity.
- Alternatively, by Picard's little theorem, since f takes all values except possibly one in any disc around z_0, e^f takes all values except possibly two in any disc around z_0, and this forces z_0 to be an essential singularity.

(f) If f has a pole at z_0, show that e^f has an essential singularity at z_0. [Hint: Use (d).]

- By (c), since f has a pole at z_0, in any open disc around z_0 there exists a sequence $\{z_n\}$ with $z_n \to z_0$ and $\text{Re}(f(z_n)) \to +\infty$ as $n \to \infty$. Then $\left|e^{f(z_n)}\right| = e^{\text{Re}(f(z_n))} \to \infty$.
- Likewise, there also exists a sequence $\{z_n\}$ with $z_n \to z_0$ and $\text{Re}(f(z_n)) \to -\infty$ as $n \to \infty$. Then $\left|e^{f(z_n)}\right| = e^{\text{Re}(f(z_n))} \to 0$.
- So e^f takes values that are both bounded and unbounded near 0, so e^f must have an essential singularity.

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7. The goal of this problem is to give another another another proof of the fundamental theorem of algebra, due to Schep. Let $p(z)$ be a nonconstant polynomial that has no roots.

(a) Show that $\dfrac{1}{zp(z)}$ has a single simple pole at $z = 0$, calculate its residue there, and show the residue is nonzero.

- Since $zp(z)$ is holomorphic, the poles of $\dfrac{1}{zp(z)}$ occur when $zp(z) = 0$ with multiplicities equal to the zero multiplicities of $zp(z)$.
- Since $p(z)$ is never zero by hypothesis, the only zero occurs at $z = 0$ and it is a zero of order 1. So $\dfrac{1}{zp(z)}$ has a single pole at $z = 0$ of order 1.
- The residue, per our results, is $\lim_{z\to 0} \dfrac{z}{zp(z)} = \lim_{z\to 0} \dfrac{1}{p(z)} = \dfrac{1}{p(0)}$, which is nonzero since $p(0)$ does not vanish.

(b) If γ_r is the counterclockwise circle $|z| = r$, show that $\int_{\gamma_r} \dfrac{1}{zp(z)}\, dz \to 0$ as $r \to \infty$. Obtain a contradiction. [Hint: Suppose $|p(z)|$ has minimum M_r on γ_r. As shown in class, $M_r \to \infty$ as $r \to \infty$.]

- Per the hint, let M_r be the minimum of $|p(z)|$ on γ_r (this minimum exists since γ_r is closed and bounded and p is continuous).
- Then $|p(z)| \geq M_r$ on γ_r, and as shown in class, $M_r \to \infty$ as $r \to \infty$. Then on γ_r, we have $\left|\dfrac{1}{zp(z)}\right| = \dfrac{1}{r} \cdot \dfrac{1}{|p(z)|} \geq \dfrac{1}{rM_r}$.
- Thus by the arclength estimate we have $\left|\int_{\gamma_r} \dfrac{1}{zp(z)}\, dz\right| \leq 2\pi r \cdot \dfrac{1}{rM_r} = \dfrac{2\pi}{M_r} \to 0$ as $r \to \infty$.
- Finally, since by (a) the function $\dfrac{1}{zp(z)}$ has a single simple pole at $z = 0$, the residue theorem yields $\int_{\gamma_r} \dfrac{1}{zp(z)}\, dz = \dfrac{2\pi i}{p(0)} \neq 0$. But this contradicts the fact that $\int_{\gamma} \dfrac{1}{zp(z)}\, dz \to 0$ for large r.

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8. [Challenge] The goal of this problem is to describe how to compute a Laurent expansion for $\csc(z)$ on the annulus $\pi < |z| < 2\pi$ starting from the Laurent expansion $\csc(z) = z^{-1} + \frac{1}{6}z + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \cdots$, which converges for $0 < |z| < \pi$.

(a) Find the terms in the Laurent expansion of $\csc(z)$ centered at $z = -\pi$ and $z = \pi$ up to degree 5. [Hint: What is $\csc(z \pm \pi)$?]

- Since $\sin(z-\pi) = -\sin(z) = \sin(z+\pi)$ using the sine addition formula, the same holds for cosecant.
- So $\csc(z) = -(z+\pi)^{-1} - \frac{1}{6}(z+\pi) - \frac{7}{360}(z+\pi)^3 - \frac{31}{15120}(z+\pi)^5 - \cdots$ centered at $-\pi$ and $\csc(z) = -(z-\pi)^{-1} - \frac{1}{6}(z-\pi) - \frac{7}{360}(z-\pi)^3 - \frac{31}{15120}(z-\pi)^5 - \cdots$ centered at π.

(b) Verify that $\csc(z) + \frac{2z}{z^2-\pi^2} = \csc(z) + \frac{1}{z-\pi} + \frac{1}{z+\pi}$ has removable singularities at $z = -\pi$ and $z = \pi$. Deduce that it has a Laurent expansion centered at $z = 0$ with radius of convergence 2π, and find the terms up to degree 5.

- From the Laurent expansions in (a) we see $\csc(z) + \frac{1}{z+\pi} = -\frac{1}{6}(z+\pi) - \frac{7}{360}(z+\pi)^3 - \cdots$ at $z = -\pi$ and $\csc(z) + \frac{1}{z-\pi} = -\frac{1}{6}(z-\pi) - \frac{7}{360}(z-\pi)^3 - \cdots$ at $z = \pi$, so since the other term is holomorphic we conclude $\csc(z) + \frac{1}{z+\pi} + \frac{1}{z-\pi}$ has removable singularities at both $z = -\pi$ and $z = \pi$.
- Then since the singularities of $\csc(z)$ occur at the zeroes of $\sin(z)$, namely $z = 0, \pm\pi, \pm 2\pi, \ldots$, the singularities of $f(z)$ are the same. Since the singularities at $z = \pm\pi$ are removable, the next closest ones are at $z = \pm 2\pi$. Thus, the Laurent expansion of $\csc(z) + \frac{1}{z+\pi} + \frac{1}{z-\pi}$ converges for $0 < |z| < 2\pi$.
- Since we already have a Laurent expansion near 0 for $\csc(z) = z^{-1} + \frac{1}{6}z + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \cdots$ and for $\frac{2z}{z^2-\pi^2} = -\frac{2z}{\pi^2} \cdot \frac{1}{1-(z^2/\pi^2)} = -\frac{2z}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{\pi^{2n}} z^{2n} = -\frac{2}{\pi^2}z - \frac{2}{\pi^4}z^3 - \frac{2}{\pi^6}z^5 - \cdots$, by uniqueness of Laurent expansions their sum must be the one we are seeking.
- So, $\csc(z) + \frac{2z}{z^2-\pi^2} = [z^{-1} + \frac{1}{6}z + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \cdots] + [-\frac{2}{\pi^2}z - \frac{2}{\pi^4}z^3 - \frac{2}{\pi^6}z^5 - \cdots] =$ $\boxed{z^{-1} + (\frac{1}{6} - \frac{2}{\pi^2})z + (\frac{7}{360} - \frac{2}{\pi^4})z^2 + (\frac{31}{15120} - \frac{2}{\pi^6})z^5 + \cdots}$.

(c) Explain why $\csc(z)$ has a Laurent expansion on the annulus $\pi < |z| < 2\pi$ and compute its terms from degree -5 to degree 5.

- Since the singularities of $\csc(z)$ occur at $z = 0, \pm\pi, \pm 2\pi, \ldots$, we see that no singularities lie in the annulus$\pi < |z| < 2\pi$ so it has a Laurent expansion there.
- By (b) we have a Laurent expansion for $\csc(z) + \frac{2z}{z^2-\pi^2}$ for $0 < |z| < 2\pi$, so we just need to subtract the Laurent expansion for $\frac{2z}{z^2-\pi^2}$ on this annulus.
- But since $\frac{2z}{z^2-\pi^2} = 2z^3 \cdot \frac{1}{1-(\pi^2/z^2)} = 2z^3\sum_{n=0}^{\infty} \pi^{2n} z^{-2n} = \cdots + 2\pi^8 z^{-5} + 2\pi^6 z^{-3} + 2\pi^4 z^{-1} + 2\pi^2 z + 2z^3$, subtracting yields the desired expansion: $\csc(z) =$ $\boxed{\cdots - 2\pi^8 z^{-5} - 2\pi^6 z^{-3} + (1-2\pi^4)z^{-1} + (\frac{1}{6} - \frac{2}{\pi^2} - 2\pi^2)z + (\frac{7}{360} - \frac{2}{\pi^4} - 2)z^3 + (\frac{31}{15120} - \frac{2}{\pi^6})z^5 + \cdots}$.
