E. Dummit's Math 4555 ~ Complex Analysis, Fall 2025 ~ Homework 9, due Fri Nov 14th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each function $f(z)$, identify (i) all zeroes and their orders, (ii) all removable singularities, (iii) all poles and their orders, and (iv) all essential singularities:

   (a) $f(z) = e^z - 1$.

   (b) $f(z) = \dfrac{1}{e^z - 1}$.

   (c) $f(z) = \dfrac{e^z - 2}{e^z - 1}$.

   (d) $f(z) = \dfrac{z^5 - z}{z^3 - 2z^2 + z}$.

   (e) $f(z) = \dfrac{\sin(z^2)}{z^3}$.

   (f) $f(z) = (z^2 - 1)\sin(\pi/z)$.

   (g) $f(z) = \dfrac{e^{\pi z}\sin(\pi z)}{z(z^2 - 1)^3}$.

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2. Calculate the residue of each function $f(z)$ at the given point:

   (a) $f(z) = e^z/\sin(z)$ at $z = 0$.

   (b) $f(z) = e^z/\sin(z)$ at $z = \pi$.

   (c) $f(z) = e^z/(z+1)^2$ at $z = -1$.

   (d) $f(z) = e^z/\sin(z^2)$ at $z = 0$.

   (e) $f(z) = \dfrac{1}{z^3(z+1)^4}$ at $z = 0$.

   (f) $f(z) = \dfrac{1}{z^3(z+1)^4}$ at $z = -1$.

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3. Consider the function $f(z) = \dfrac{e^{1/z}}{1 - z^2}$ and observe that $f(z)$ has an essential singularity at $z = 0$.

   (a) Identify the poles for $f(z)$ and compute the order and residue at each pole.

   (b) Find the terms from order -4 to 0 of the Laurent expansion for $f(z)$ on the region $|z| > 1$.

   (c) Find the terms from order -2 to 2 of the Laurent expansion for $f(z)$ on the region $0 < |z| < 1$. [The coefficients are infinite sums: please express them in terms of sinh and cosh.]

   (d) Find $\int_\gamma f(z)\,dz$ where γ is the counterclockwise circle $|z| = 0.1$.

   (e) Find $\int_\gamma f(z)\,dz$ where γ is the counterclockwise circle $|z - 1| = 0.1$.

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4. Calculate the following contour integrals:

   (a) $\int_\gamma \dfrac{1}{z^2 - 2025z}\,dz$ where γ is the counterclockwise circle $|z| = 1$.

   (b) $\int_\gamma \dfrac{1}{z^2 - 2025z}\,dz$ where γ is the counterclockwise circle $|z| = 5000$.

   (c) $\int_\gamma \dfrac{e^z}{z(z-1)(z-2)}\,dz$ where γ is the counterclockwise circle $|z| = 3$.

   (d) $\int_\gamma \dfrac{1}{e^z - 1}\,dz$ where γ is the counterclockwise boundary of the square with vertices $(\pm\frac{5}{2} \pm \frac{5}{2}i)\pi$.

   (e) $\int_\gamma \dfrac{\sin \pi z}{\sin z}\,dz$ where γ is the counterclockwise circle $|z - 1| = 4$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose that $f(z)$ is bounded and holomorphic on $\mathbb{C}\backslash\{0\}$. Show that $f(z)$ is constant.

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6. Suppose $f(z)$ is meromorphic on a region R and that f has an isolated singularity at $z = z_0$.

   (a) If $\lim_{z\to z_0}(z - z_0)f(z) = 0$, show that f has a removable singularity at z_0.

   (b) If f has a zero of order k at z_0, show $f'(z)/f(z)$ has a simple pole at z_0 with residue k.

   (c) If f has a pole of order k at z_0, show $f'(z)/f(z)$ has a simple pole at z_0 with residue $-k$.

   (d) If f has a pole at z_0, show that $\text{Re}(f)$ and $\text{Im}(f)$ take arbitrarily large positive and negative values as $z \to z_0$. [Hint: If $f(z) = re^{i\theta}(z - z_0)^{-k} + \cdots$, take $z = z_0 + t^{1/k}e^{i\theta/k}$ as $t \to 0$ in different directions.]

   (e) If f has an essential singularity at z_0, show that e^f also has an essential singularity at z_0.

   (f) If f has a pole at z_0, show that e^f has an essential singularity at z_0. [Hint: Use (d).]

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7. The goal of this problem is to give another another another proof of the fundamental theorem of algebra, due to Schep. Let $p(z)$ be a nonconstant polynomial that has no roots.

   (a) Show that $\dfrac{1}{zp(z)}$ has a single simple pole at $z = 0$, calculate its residue there, and show the residue is nonzero.

   (b) If γ_r is the counterclockwise circle $|z| = r$, show that $\int_{\gamma_r} \dfrac{1}{zp(z)}\,dz \to 0$ as $r \to \infty$. Obtain a contradiction. [Hint: Suppose $|p(z)|$ has minimum M_r on γ_r. As shown in class, $M_r \to \infty$ as $r \to \infty$.]

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8. [Challenge] The goal of this problem is to describe how to compute a Laurent expansion for $\csc(z)$ on the annulus $\pi < |z| < 2\pi$ starting from the Laurent expansion $\csc(z) = z^{-1} + \dfrac{1}{6}z + \dfrac{7}{360}z^3 + \dfrac{31}{15120}z^5 + \cdots$, which converges for $0 < |z| < \pi$.

   (a) Find the terms in the Laurent expansion of $\csc(z)$ centered at $z = -\pi$ and $z = \pi$ up to degree 5. [Hint: What is $\csc(z \pm \pi)$?]

   (b) Verify that $\csc(z) + \dfrac{2z}{z^2 - \pi^2} = \csc(z) + \dfrac{1}{z - \pi} + \dfrac{1}{z + \pi}$ has removable singularities at $z = -\pi$ and $z = \pi$. Deduce that it has a Laurent expansion centered at $z = 0$ with radius of convergence 2π, and find the terms up to degree 5.

   (c) Explain why $\csc(z)$ has a Laurent expansion on the annulus $\pi < |z| < 2\pi$ and compute its terms from degree -5 to degree 5.
