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1. Find the radius of convergence for each power series centered at the given point:

   (a) The series expansion of $1/(z-2)$ centered at $z=0$.

   - This function is holomorphic for $z \neq 2$, so the maximal radius centered at $z=0$ is the distance to the nearest non-holomorphic point, which is $\boxed{2}$.

   (b) The series expansion of $z^3/(z^2+1)^2$ centered at $z=1$.

   - This function is holomorphic for $z^2 \neq 1$, which is to say $z \neq 1, -1$, so the maximal radius centered at $z=0$ is the distance to the nearest non-holomorphic point, which is $\boxed{\sqrt{2}}$.

   (c) The series expansion of $\sec z$ centered at $z=0$.

   - This function is holomorphic for $\cos z \neq 0$, which is to say for $z \neq \pi/2 + k\pi$ for integers k. So the maximal radius is $\boxed{\pi/2}$.

   (d) The series expansion of $\dfrac{z}{\sin z}$ centered at $z=0$.

   - The function $\dfrac{z}{\sin z}$ has a removable singularity at $z=0$ (as is seen by writing down the power series there) and the denominator is zero for $z = k\pi$ for integers k. So the maximal radius is $\boxed{\pi}$.

   (e) The series expansion of $\text{Log}(z)$ centered at $z = 1+i$.

   - This function is $\text{Log}(z)$ which is holomorphic for $z \notin [0,\infty)$. So the maximal radius centered at $z = 1+i$ is the distance to the nearest non-holomorphic point 1, so the radius is $\boxed{1}$.

   (f) The series expansion of $\dfrac{1}{e^{1/z}-1}$ centered at $z=i$.

   - This function is holomorphic for $z \neq 0$ and $e^{1/z} \neq 1$, which is to say for $z \neq 0$ and $z \neq 1/(2\pi k i)$ for integers k.
   - The closest such point to $z=i$ is $z = i/(2\pi)$. So the maximal radius centered at $z=i$ is the distance to this point, which is $\boxed{1-1/(2\pi)}$.

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2. Solve the following optimization problems, and briefly justify your responses:

   (a) Find the maximum value of $\left|z^2+3z-1\right|$ for $|z| \leq 1$.

   - Since $f(z) = z^2+3z-1$ is holomorphic, by the maximum modulus principle the maximum occurs on the boundary circle $|z|=1$.
   - For $z = e^{i\theta}$ we see $\left|z^2+3z-1\right|^2 = (e^{2i\theta}+3e^{i\theta}-1)(e^{-2i\theta}+3e^{-i\theta}-1) = 11-(e^{2i\theta}+e^{-2i\theta}) = 11-2\cos 2\theta$.
   - The maximum value clearly occurs for $\cos 2\theta = -1$, which is to say for $\theta = \pi/2, 3\pi/2$, yielding $z = \pm i$, and the maximum is $\boxed{\sqrt{13}}$.

   (b) Find the maximum value of $\left|z^2+i\right|$ for $|z| \leq 2$.

   - Since $f(z) = z^2+i$ is holomorphic, by the maximum modulus principle the maximum occurs on the boundary circle $|z|=2$.
   - For $z = 2e^{i\theta}$ we see $\left|z^2+i\right|^2 = (4e^{2i\theta}+i)(4e^{-2i\theta}-i) = 17+4ie^{-2i\theta}-4ie^{2i\theta} = 17+8\sin 2\theta$.
   - The maximum value clearly occurs for $\sin 2\theta = 1$, which is to say for $\theta = \pi/4, 9\pi/4$ yielding $z = \pm(\sqrt{2}+i\sqrt{2})$, and the maximum is $\boxed{\sqrt{25}=5}$.

   (c) Find the maximum value of $\left|20z^{25}+3+4i\right|$ for $|z| \leq 1$. [Hint: Triangle inequality.]

   - By the triangle inequality we have $\left|20z^{25}+3+4i\right| \leq \left|20z^{25}\right| + |3+4i| = 20+5 = 25$.

- In fact, this bound is achievable: per the maximum modulus principle we want to look only when $|z| = 1$, and if we do so, then to get equality in the triangle inequality we want to take $20z^{25}$ to be a nonnegative real multiple of $3 + 4i$. This can be done if we take $z^{25} = \frac{3}{5} + \frac{4}{5}i$ (using any 25th root). So the maximum is in fact $\boxed{25}$.

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3. Find the requested terms in the Laurent expansion for each function $f(z)$ on the given region:

   (a) The terms from degree -5 to 5 of $f(z) = 1/(z + z^2)$ for $0 < |z| < 1$.

   - We have $f(z) = \frac{1}{z} \cdot \frac{1}{1+z} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^n = \boxed{z^{-1} - 1 + z - z^2 + z^3 - z^4 + z^5 + \cdots}$.

   (b) The terms from degree -5 to 5 of $f(z) = 1/(z + z^2)$ for $|z| > 1$.

   - We have $f(z) = \frac{1}{z^2} \cdot \frac{1}{1 + 1/z} = \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n z^{-n} = \boxed{\cdots - z^{-5} + z^{-4} - z^{-3} + z^{-2}}$.

   (c) The terms from degree -3 to 3 of $f(z) = 1/(z + z^2)$ for $0 < |z + 1| < 1$.

   - With $w = z+1$, $f(z) = -\frac{1}{w} \cdot \frac{1}{1-w} = -\frac{1}{w} \sum_{n=0}^{\infty} w^n = \boxed{-(z+1)^{-1} - 1 - (z+1)^1 - (z+1)^2 - (z+1)^3 - \cdots}$.

   (d) The terms from degree -4 to 4 of $f(z) = 1/(z + z^2)$ for $|z + 1| > 1$.

   - With $w = z + 1$, $f(z) = \frac{1}{w^2} \cdot \frac{1}{1 - 1/w} = \frac{1}{w^2} \sum_{n=0}^{\infty} w^{-n} = \boxed{\cdots + (z+1)^{-4} + (z+1)^{-3} + (z+1)^{-2}}$.

   (e) The terms from degree -3 to 3 of $f(z) = \dfrac{1}{e^z - 1}$ for $0 < |z| < 2\pi$.

   - We compute that $f(z) = \dfrac{1}{z + z^2/2! + z^3/3! + z^4/4! \cdots} = z^{-1} \cdot \dfrac{1}{1 + z/2 + z^2/6 + z^3/24 + \cdots}$ $= z^{-1}[1 - \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \cdots] = \boxed{z^{-1} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots}$.

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4. Prove that the function $f(z)$ is entire if and only if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\lim_{n\to\infty} |a_n|^{1/n} = 0$.

   - If $f(z)$ is entire, then it is analytic at 0 and its power series there has infinite radius of convergence: this means $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\limsup_{n\to\infty} |a_n|^{1/n} = 0$.
   - But the limsup is zero if and only if the limit itself exists and is zero. So $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\lim_{n\to\infty} |a_n|^{1/n} = 0$.
   - Conversely, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\lim_{n\to\infty} |a_n|^{1/n} = 0$, then f is analytic at 0 with infinite radius of convergence. Since analytic functions are holomorphic inside their radius of convergence, that means f is entire.

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5. The goal of this problem is to prove the minimum modulus principle.

   (a) Suppose that $f(z)$ is holomorphic in a closed bounded region R and $|f(z)| > 0$ on R. Show that if the minimum value of $|f(z)|$ occurs at a point z_0 in the interior of R, then f is constant on R. [Hint: Consider $1/f$.]

   - Consider $1/f(z)$: since $f(z)$ is nonzero on R, it is holomorphic on R. Then by the maximum modulus principle applied to $1/f(z)$, if the maximum modulus of $1/f(z)$ occurs at a point z_0 in the interior of R, then $1/f$ is constant on R.
   - Taking reciprocals everywhere yields the desired statement, since $|1/f(z)|$ is maximized precisely when $|f(z)|$ is minimized.

   (b) Deduce that if $f(z)$ is holomorphic in a closed bounded region R and $|f(z)| > 0$ on R, then the minimum value of $|f(z)|$ on R must occur at a point on the boundary of R.

- Take the contrapositive of (a).

(c) Show that the hypothesis $|f(z)| > 0$ cannot be removed from part (b) by giving an example of a nonconstant holomorphic $f(z)$ such that the minimum value of $|f(z)|$ occurs at a point z_0 in the interior of R.

- There are lots of options but an easy one is $f(z) = z$ with $R : |z| \le 1$. Then $f(0) = 0$ while $|f(z)| = 1$ on the boundary of R, so the minimum of $|f(z)|$ occurs only at the interior point $z_0 = 0$.

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6. The goal of this problem is to give another proof of the fundamental theorem of algebra, due to Boas. Suppose that $p(z) = \sum_{n=0}^{k} a_n z^n$ is a polynomial of degree $k \ge 1$ that is never zero.

(a) Define $q(z) = p(z)\overline{p}(z) = [\sum_{n=0}^{k} a_n z^n][\sum_{n=0}^{k} \overline{a_n} z^n]$. Show that $q(z)$ has degree $2k \ge 2$, has real coefficients, and is never zero on $\mathbb{R}$ hence is either always positive or always negative on $\mathbb{R}$. [Hint: Notice that $\overline{p}(z) = \overline{p(\overline{z})}$.]

- First, the leading term is $|a_k|^2 z^{2k}$ so the degree is $2k \ge 2$.
- Next, per the hint we first note $\overline{p}(z) = \overline{p(\overline{z})}$: thus conjugating the coefficients of a polynomial is the same as evaluating at on $\overline{z}$ and then conjugating the result.
- So, conjugating the coefficients of q yields $\overline{q}(z) = \overline{q(\overline{z})} = \overline{p(\overline{z})\overline{p}(\overline{z})} = \overline{p}(z)p(z) = q(z)$. Since this just gives $q(z)$ again, that means all coefficients of q are real.
- Next, $q(z) = 0$ implies $p(z)\overline{p}(z) = 0$ which is the same as $p(z)\overline{p(\overline{z})} = 0$. So either $p(z) = 0$ which is impossible or $\overline{p(\overline{z})} = 0$ whence $p(\overline{z}) = 0$, but this is again impossible. So q is never zero.
- Finally since q is never zero on $\mathbb{R}$ and has real coefficients, by the intermediate value theorem it cannot change sign, so it is either always positive or always negative.

(b) Continuing (a), let $r(z) = z^{2k} q(z + z^{-1})$. Show that $r(z)$ is entire and nonzero.

- By (a), q is a polynomial of degree $2k$ so suppose $q(z) = b_0 + b_1 z + \cdots + b_{2k} z^{2k}$.
- Then $r(z) = z^{2k} q(z + z^{-1}) = z^{2k}[b_0 + b_1(z + z^{-1}) + \cdots + b_{2k}(z + z^{-1})^{2k}] = z^{2k} b_0 + z^{2k-1} b_1 (z^2 + 1) + \cdots + b_{2k}(z^2 + 1)^{2k}$ which is a polynomial (technically, depending on one's philosophy, we are ignoring the removable singularity at $z = 0$).
- Clearly $r(z) = 0$ can be zero only when $z^{2k} = 0$ or when $q(z + z^{-1}) = 0$. Since q is never zero the latter cannot happen, and since $r(0) = b_{2k} \ne 0$ from the expansion above we see that $r(z)$ is never zero.

(c) Continuing (b), show that $-i \int_\gamma \dfrac{z^{2k-1}}{r(z)}\, dz = \int_0^{2\pi} \dfrac{1}{q(2\cos\theta)}\, d\theta$ where γ is the counterclockwise boundary of the unit circle. Explain why the first integral is zero while the second integral is nonzero, and obtain a contradiction.

- Starting with $-i \int_\gamma \dfrac{z^{2k-1}}{r(z)}\, dz$ take the parametrization $\gamma(t) = e^{i\theta}$ for $0 \le \theta \le 2\pi$.
- Then $r(z) = r(e^{i\theta}) = (e^{i\theta})^{2k} q(e^{i\theta} + e^{-i\theta}) = e^{2ki\theta} q(2\cos\theta)$ with $z^{2k-1} dz = (e^{i\theta})^{2k-1} i e^{i\theta} d\theta = i e^{2ki\theta}\, d\theta$.
- So making the substitution yields $-i \int_\gamma \dfrac{z^{2k-1}}{r(z)}\, dz = -i \int_0^{2\pi} \dfrac{i e^{2ki\theta}}{e^{2ki\theta} q(2\cos\theta)}\, d\theta = \int_0^{2\pi} \dfrac{1}{q(2\cos\theta)}\, d\theta$. So we get the claimed equality.
- For the first integral, we see that $\dfrac{z^{2k-1}}{r(z)}$ is holomorphic everywhere since $r(z)$ is holomorphic and nonzero by (b), so by our results, it integrates to zero on the closed contour γ.
- On the other hand, by (a), the expression $q(2\cos\theta)$ is always either positive or negative on $[0, 2\pi]$, so the integral is nonzero. This is the desired contradiction, since zero cannot equal a nonzero number.

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7. The goal of this problem is to give another proof of the differentiation-via-integration formula $f'(z_0) = \dfrac{1}{2\pi i} \int_\gamma \dfrac{f(z)}{(z - z_0)^2}\, dz$. So suppose f is a holomorphic function on a simply connected region R and let γ be a counterclockwise circle of radius $r > 0$ centered at z_0 in the interior of R such that the disc $|z - z_0| \le r$ lies inside R.

(a) Show that $\dfrac{f(z_0+h)-f(z_0)}{h} = \dfrac{1}{2\pi i}\int_\gamma \dfrac{f(z)}{(z-z_0)(z-z_0-h)}\,dz$. [Hint: Use Cauchy's integral formula.]

- By Cauchy's integral formula we have $f(z_0) = \dfrac{1}{2\pi i}\int_\gamma \dfrac{f(z)}{z-z_0}\,dz$ and $f(z_0+h) = \dfrac{1}{2\pi i}\int_\gamma \dfrac{f(z)}{z-(z_0+h)}\,dz$.
- Then $\dfrac{f(z_0+h)-f(z_0)}{h} = \dfrac{1}{2\pi i}\int_\gamma \dfrac{1}{h}\left[\dfrac{f(z)}{z-z_0-h} - \dfrac{f(z)}{z-z_0}\right]dz = \dfrac{1}{2\pi i}\int_\gamma \dfrac{f(z)}{(z-z_0)(z-z_0-h)}\,dz$ by putting the fractions together and cancelling the factor of h that drops out.

(b) [Challenge] For $|h| < r$, let $g_h(z) = \dfrac{f(z)}{(z-z_0)(z-z_0-h)}$. Show that as $h \to 0$ the functions $g_h(z)$ converge uniformly to the limit $g(z) = \dfrac{f(z)}{(z-z_0)^2}$. [Hint: Restrict attention to $|h| < r/2$, then suppose $|f(z)| \le M$ on γ_r and bound $|g_h(z) - g(z)|$ from above.]

- As noted several times in class, since f is continuous and γ_r is closed, f is bounded on γ_r, so suppose that $|f(z)| \le M$ and let $\epsilon > 0$.
- Then for $|h| < \min(r/2, \dfrac{r^3\epsilon}{2M})$, we have $|g_h(z) - g(z)| = \left|\dfrac{f(z)\cdot h}{(z-z_0)^2(z-z_0-h)}\right| < \dfrac{M\cdot(r^3\epsilon/(2M))}{r^2(r/2)} = \epsilon$.

(c) Show that $f'(z_0) = \dfrac{1}{2\pi i}\int_\gamma \dfrac{f(z)}{(z-z_0)^2}\,dz$. [Hint: Use uniform convergence to change the order of the integral and the limit as $h \to 0$.]

- Since the functions g_h converge uniformly to g as $h \to 0$, we may interchange the limit and integral to obtain

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0}\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)(z-z_0-h)}\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \lim_{h\to 0}\frac{f(z)}{(z-z_0)(z-z_0-h)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^2}\,dz \end{aligned}$$

as required.

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8. [Challenge] The goal of this problem is to give another another proof of the fundamental theorem of algebra that does not require any actual complex analysis. Suppose $p(z)$ is a polynomial.

(a) Show that $|p(z)|$ must attain its minimum value at some point in $\mathbb{C}$. [Hint: Since $\lim_{|z|\to\infty}|p(z)| = \infty$, pick R with $|p(z)| > |p(0)|$ for $|z| > R$. Then use the extreme value theorem on the region $|z| \le R$.]

- As shown in class, we have $\lim_{|z|\to\infty}|p(z)| = \infty$. So in particular there exists some R such that $|p(z)| > |p(0)|$ for $|z| > R$.
- Now, the region $|z| \le R$ is closed and bounded and $p(z)$ is a polynomial hence continuous, hence $|p(z)|$ is also continuous. By the extreme value theorem, a continuous function on a closed bounded region attains its minimum and maximum, so there exists some z_0 with $|p(z_0)| \le |p(z)|$ for all $|z| \le R$.
- In particular we have $|p(z_0)| \le |p(0)| < |p(z)|$ for all $|z| > R$, so in fact $|p(z_0)| \le |p(z)|$ for all $z \in \mathbb{C}$. This means $|p(z)|$ attains its global minimum value at z_0.

(b) Suppose that $q(z) = 1 + b(z-z_0)^r + \sum_{n=r+1}^k b_n(z-z_0)^n$ where $b \ne 0$. Show that there exists z with $|q(z)| < 1$. [Hint: Take $b(z-z_0)^r = -t$ and then show the sum is small relative to t as $t \to 0+$.]

- Following the hint, restrict attention to the situation where $b(z-z_0)^r = -t$ is a small positive real number, which is to say, $z = z_0 + (-t/b)^{1/r}$ as $t \to 0+$ from below.
- Then $q(z) = 1 - t + \sum_{n=r+1}^k b_n(-t/b)^{n/r} = 1 - t + \sum_{n=r+1}^k c_n t^{n/r}$ where $c_n = b_n/(-b)^{n/r}$.
- Since $\lim_{t\to 0}\sum_{n=r+1}^k c_n t^{(n-r)/r} = 0$ since all of the terms have a positive power of t, in particular there exists $\delta > 0$ such that for $0 < t < \delta$ we have $\left|\sum_{n=r+1}^k c_n t^{n/r}\right| < t/2$.
- Then for such t, by the triangle inequality we have $|q(z)| \le |1-t| + \left|\sum_{n=r+1}^k c_n t^{n/r}\right| \le 1 - t/2 < 1$. Thus $q(z)$ takes a value of absolute value less than 1 as claimed.

(c) Suppose that $p(z) = \sum_{n=0}^{k} a_n(z - z_0)^n$ is not constant and $|p(z_0)| > 0$. Show that there exists some z with $|p(z)| < |p(z_0)|$. [Hint: Write $p(z)/a_0 = 1 + b(z - z_0)^r + \sum_{n=r+1}^{k} b_n(z - z_0)^n$.]

- Per the hint since $a_0 = p(z_0)$ is not zero, we may divide through by a_0 and write $p(z)/a_0 = 1+\sum_{n=1}^{k}(a_n/a_0)(z-z_0)^n$. Now since p is not constant there must be some nonzero coefficient a_n/a_0 in the sum: suppose the one with smallest n is a_r/a_0.
- Then for $b = a_r/a_0$ and $b_n = a_n/a_0$ we have $p(z)/a_0 = 1 + b(z - z_0)^r + \sum_{n=r+1}^{k} b_n(z - z_0)^n$. By (b) applied to this $q(z)$ there exists some z with $|q(z)| < 1$, which is to say, $|p(z)| < |a_0| = |p(z_0)|$, as desired.

(d) Show that the minimum value of $|p(z)|$ must be zero and deduce that $p(z)$ has a root in $\mathbb{C}$.

- By (a), $|p(z)|$ attains its minimum value at some point z_0. If this minimum is not zero, then by (c), there exists some z with $|p(z)| < |p(z_0)|$. This is a contradiction. Hence the minimum must be zero, so $p(z)$ has a root in $\mathbb{C}$.
