E. Dummit's Math 4555 $\sim$ Complex Analysis, Fall 2025 $\sim$ Homework 8, due Fri Nov 7th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the radius of convergence for each power series centered at the given point:
   (a) The series expansion of $1/(z-2)$ centered at $z=0$.
   (b) The series expansion of $z^3/(z^2+1)^2$ centered at $z=1$.
   (c) The series expansion of $\sec z$ centered at $z=0$.
   (d) The series expansion of $\dfrac{z}{\sin z}$ centered at $z=0$.
   (e) The series expansion of $\text{Log}(z)$ centered at $z=1+i$.
   (f) The series expansion of $\dfrac{1}{e^{1/z}-1}$ centered at $z=i$.

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2. Solve the following optimization problems, and briefly justify your responses:
   (a) Find the maximum value of $\left|z^2+3z-1\right|$ for $|z| \le 1$.
   (b) Find the maximum value of $\left|z^2+i\right|$ for $|z| \le 2$.
   (c) Find the maximum value of $\left|20z^{25}+3+4i\right|$ for $|z| \le 1$. [Hint: Triangle inequality.]

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3. Find the requested terms in the Laurent expansion for each function $f(z)$ on the given region:
   (a) The terms from degree -5 to 5 of $f(z) = 1/(z+z^2)$ for $0 < |z| < 1$.
   (b) The terms from degree -5 to 5 of $f(z) = 1/(z+z^2)$ for $|z| > 1$.
   (c) The terms from degree -3 to 3 of $f(z) = 1/(z+z^2)$ for $0 < |z+1| < 1$.
   (d) The terms from degree -4 to 4 of $f(z) = 1/(z+z^2)$ for $|z+1| > 1$.
   (e) The terms from degree -3 to 3 of $f(z) = \dfrac{1}{e^z-1}$ for $0 < |z| < 2\pi$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Prove that the function $f(z)$ is entire if and only if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\lim_{n\to\infty} |a_n|^{1/n} = 0$.

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5. The goal of this problem is to prove the minimum modulus principle.
   (a) Suppose that $f(z)$ is holomorphic in a closed bounded connected region R and $|f(z)| > 0$ on R. Show that if the minimum value of $|f(z)|$ occurs at a point z_0 in the interior of R, then f is constant on R. [Hint: Consider $1/f$.]
   (b) Deduce that if $f(z)$ is holomorphic in a closed bounded connected region R and $|f(z)| > 0$ on R, then the minimum value of $|f(z)|$ on R must occur at a point on the boundary of R.
   (c) Show that the hypothesis $|f(z)| > 0$ cannot be removed from part (b) by giving an example of a nonconstant holomorphic $f(z)$ such that the minimum of $|f(z)|$ does not occur on the boundary of R.

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6. The goal of this problem is to give another proof of the fundamental theorem of algebra, due to Boas. Suppose that $p(z) = \sum_{n=0}^{k} a_n z^n$ is a polynomial of degree $k \geq 1$ that is never zero.

   (a) Define $q(z) = p(z)\overline{p}(z) = [\sum_{n=0}^{k} a_n z^n][\sum_{n=0}^{k} \overline{a_n} z^n]$. Show that $q(z)$ has degree $2k \geq 2$, has real coefficients, and is never zero on $\mathbb{R}$ hence is either always positive or always negative on $\mathbb{R}$. [Hint: Notice that $\overline{p}(z) = \overline{p(\overline{z})}$.]

   (b) Continuing (a), let $r(z) = z^{2k} q(z + z^{-1})$. Show that $r(z)$ is entire and nonzero.

   (c) Continuing (b), show that $-i \int_\gamma \frac{z^{2k-1}}{r(z)}\, dz = \int_0^{2\pi} \frac{1}{q(2\cos\theta)}\, d\theta$ where γ is the counterclockwise boundary of the unit circle. Explain why the first integral is zero while the second integral is nonzero, and obtain a contradiction.

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7. The goal of this problem is to give another proof of the differentiation-via-integration formula $f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^2}\, dz$. So suppose f is a holomorphic function on a simply connected region R and let γ be a counterclockwise circle of radius $r > 0$ centered at z_0 in the interior of R such that the disc $|z - z_0| \leq r$ lies inside R.

   (a) Show that $\frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)(z - z_0 - h)}\, dz$. [Hint: Use Cauchy's integral formula.]

   (b) [Challenge] For $|h| < r$, let $g_h(z) = \frac{f(z)}{(z - z_0)(z - z_0 - h)}$. Show that as $h \to 0$ the functions $g_h(z)$ converge uniformly to the limit $g(z) = \frac{f(z)}{(z - z_0)^2}$. [Hint: Restrict attention to $|h| < r/2$, then suppose $|f(z)| \leq M$ on γ_r and bound $|g_h(z) - g(z)|$ from above.]

   (c) Show that $f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^2}\, dz$. [Hint: Use uniform convergence to change the order of the integral and the limit as $h \to 0$.]

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8. [Challenge] The goal of this problem is to give another another proof of the fundamental theorem of algebra that does not require any actual complex analysis. Suppose $p(z)$ is a polynomial.

   (a) Show that $|p(z)|$ must attain its minimum value at some point in $\mathbb{C}$. [Hint: Since $\lim_{|z| \to \infty} |p(z)| = \infty$, pick R with $|p(z)| > |p(0)|$ for $|z| > R$. Then use the extreme value theorem on the region $|z| \leq R$.]

   (b) Suppose that $q(z) = 1 + b(z - z_0)^r + \sum_{n=r+1}^{k} b_n (z - z_0)^n$ where $b \neq 0$. Show that there exists z with $|q(z)| < 1$. [Hint: Take $b(z - z_0)^r = -t$ and then show the sum is small relative to t as $t \to 0+$.]

   (c) Suppose that $p(z) = \sum_{n=0}^{k} a_n (z - z_0)^n$ is not constant and $|p(z_0)| > 0$. Show that there exists some z with $|p(z)| < |p(z_0)|$. [Hint: Write $p(z)/a_0 = 1 + b(z - z_0)^r + \sum_{n=r+1}^{k} b_n (z - z_0)^n$.]

   (d) Show that the minimum value of $|p(z)|$ must be zero and deduce that $p(z)$ has a root in $\mathbb{C}$.
