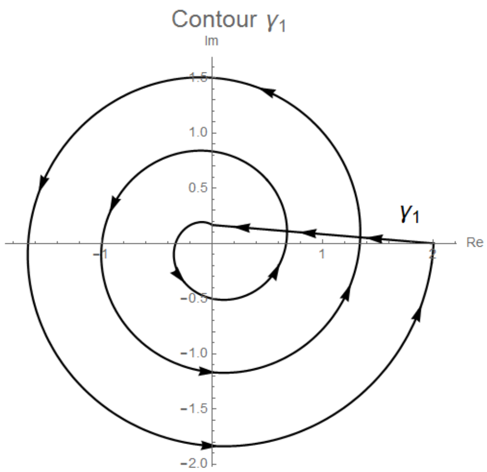
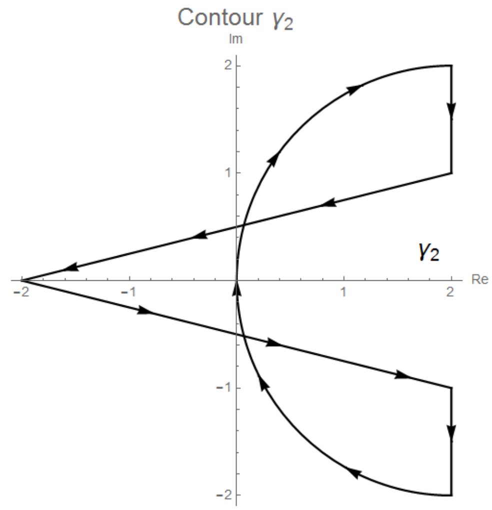
E. Dummit's Math 4555 ~ Complex Analysis, Fall 2025 ~ Homework 7, due Fri Oct 24th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let γ_1 and γ_2 be the two contours plotted below. Find the requested quantities:





   (a) The winding numbers of γ_1 around 0, 1, i, $-i$, $1+i$, $2+i$, and $-2i$.
   (b) The winding numbers of γ_2 around -1, 1, $1+i$, $1-i$, and $-1+i$.
   (c) The values of $\int_{\gamma_1} \frac{1}{z-1}\,dz$ and $\int_{\gamma_2} \frac{1}{z-1}\,dz$.
   (d) The values of $\int_{\gamma_1} \frac{1}{z-(1+i)}\,dz$ and $\int_{\gamma_2} \frac{1}{z-(1+i)}\,dz$.
   (e) The values of $\int_{\gamma_1} \frac{e^z}{z-1}\,dz$ and $\int_{\gamma_2} \frac{e^z}{z-1}\,dz$.
   (f) The value of $\int_{\gamma_2} \frac{2z}{z^2-1}\,dz$.
   (g) The value of $\int_{\gamma_2} \frac{2}{z^2-2z+2}\,dz$.

---

2. For each function f on each contour γ, calculate $\int_\gamma f(z)\,dz$:

   (a) $f(z) = \frac{z^3}{z+1}$ on the counterclockwise boundary of the circle $|z| = 4$.
   (b) $f(z) = \frac{e^z}{z-2}$ on the counterclockwise boundary of the circle $|z| = 4$.
   (c) $f(z) = \frac{\sin^2(2z)}{z-1}$ on the counterclockwise boundary of the circle $|z| = 4$.
   (d) $f(z) = \frac{z^2+1}{z^2-1}$ on the counterclockwise boundary of the rectangle with vertices ± 20, $\pm 25i$.
   (e) $f(z) = \frac{z^2+1}{z^2-1}$ on the counterclockwise boundary of the square with vertices 0, $10-10i$, 20, $10+10i$.
   (f) $f(z) = \frac{e^z}{z \sin z}$ on the counterclockwise boundary of the circle $|z-\pi| = e^{-\pi}$.
   (g) $f(z) = \overline{z}e^{1/\overline{z}}$ on the counterclockwise boundary of the circle $|z| = 2$. [Hint: On γ, $\overline{z}$ can be written in terms of z.]

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. If γ is the unit circle traversed once counterclockwise, let $I(a,b) = \int_\gamma \frac{1}{(z-a)(z-b)}\, dz$.

   (a) Find $I(a,a)$ if $|a| \neq 1$.

   (b) Find $I(a,b)$ if $|a| < 1$ and $|b| < 1$ and $a \neq b$.

   (c) Find $I(a,b)$ if $|a| < 1$ and $|b| > 1$.

   (d) Find $I(a,b)$ if $|a| > 1$ and $|b| > 1$ and $a \neq b$.

---

4. The goal of this problem is to give another way to evaluate the integral $I_\gamma(z_0) = \int_\gamma \frac{1}{z-z_0}\, dz$ where γ is any counterclockwise-oriented circle not containing z_0, which was the subject of problem 5 of homework 6.

   (a) Suppose z_0 is a distance $R > 0$ away from the closest point on the circle. Show that $\left| \int_\gamma \frac{1}{z-z_0}\, dz \right| \leq \frac{2\pi r}{R}$ where r is the radius of the circle. [Hint: Bound the integral by the arclength times the maximum of the function.]

   (b) Suppose z_0 is outside the circle. Show that $\int_\gamma \frac{1}{z-z_0}\, dz = 0$. [Hint: Move γ far away from z_0.]

   (c) Suppose z_0 is in the interior of the circle. Show that $\int_\gamma \frac{1}{z-z_0}\, dz = 2\pi i$. [Hint: Recenter γ at z_0.]

---

5. Suppose $p(z)$ is a polynomial of degree at least 2. Then $p(z)$ has finitely many complex zeroes, so they are all contained in a disc $|z| < r$ for some r. For $R \geq r$, let $I(R) = \int_{\gamma_R} \frac{1}{p(z)}\, dz$ where γ_R is the circle $|z| = R$ traversed once counterclockwise.

   (a) Show that $I(R) = I(r)$ for all $R \geq r$.

   (b) Show that $|I(R)| \leq \frac{2\pi R}{|a|\,(R-r)^d}$ where p has degree d and leading coefficient a. [Hint: You may assume $p(z)$ factors as $p(z) = a(z-z_1)(z-z_2)\cdots(z-z_d)$.]

   (c) Show that $\lim_{R\to\infty} I(R) = 0$. Deduce that $I(r) = 0$.

---

6. The goal of this problem is to give a third approach for evaluating the integral $I_\gamma(z_0) = \int_\gamma \frac{1}{z-z_0}\, dz$ where γ is any counterclockwise-oriented circle not containing z_0.

   (a) Suppose z_0 is outside the circle. Show that $\int_\gamma \frac{1}{z-z_0}\, dz = 0$. [Hint: Pick a branch of $\log(z-z_0)$ that does not intersect the circle.]

   (b) Suppose z_0 is in the interior of the circle and let the horizontal ray $z = z_0 + t$ for $t \geq 0$ intersect the circle at P. Choose any α and β on the circle such that the points P, α, β are in counterclockwise order around the circle, and take $\tilde{\gamma}$ to be the counterclockwise arc from α to β. Show that $\int_{\tilde{\gamma}} \frac{1}{z-z_0}\, dz = \operatorname{Log}(\beta - z_0) - \operatorname{Log}(\alpha - z_0)$.

   (c) Suppose z_0 is in the interior of the circle. Show that $\int_\gamma \frac{1}{z-z_0}\, dz = 2\pi i$. [Hint: In (b), let α approach P from above and β approach P from below.]

---

7. [Challenge] The goal of this problem is to establish some basic facts about the gamma and zeta functions, which are two special functions with broad utility in complex analysis, number theory, statistics, physics, and various other areas. Recall that for a positive real number α and a complex number z, we have $\alpha^z = e^{z \ln \alpha}$.

   (a) Suppose n is a positive integer. Show that $\int_0^\infty t^{n-1} e^{-t}\, dt = (n-1)!$.

   (b) Suppose that $\mathrm{Re}(z) > 1$. Show that the integral $\int_0^\infty t^{z-1} e^{-t}\, dt$ converges absolutely.

   (c) Let R be the region with $\mathrm{Re}(z) > 1$. Show that $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ is holomorphic on R. [Hint: Differentiate under the integral sign.]

   (d) Suppose that $\mathrm{Re}(z) > 1$. Show that the series $\sum_{n=1}^\infty \frac{1}{n^z}$ converges absolutely.

   (e) Let R be the region with $\mathrm{Re}(z) > 1$. Show that $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$ is holomorphic on R.
