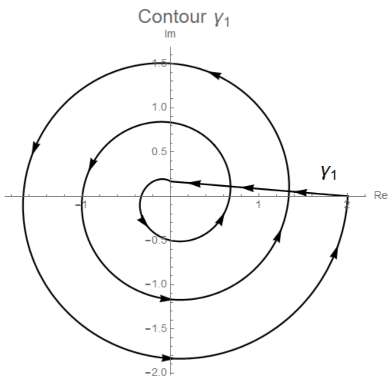
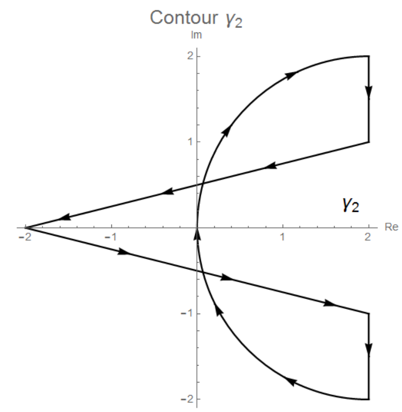
1. Let γ_1 and γ_2 be the two contours plotted below. Find the requested quantities:





(a) The winding numbers of γ_1 around 0, 1, i, $-i$, $1+i$, $2+i$, and $-2i$.

- By deforming the contour without moving it through the given point, or alternatively drawing a ray from the point to ∞ and counting the number of signed crossings right-to-left minus left-to-right (as one walks from the point to ∞ along the ray), we see the desired winding numbers are $\boxed{3}$, $\boxed{2}$, $\boxed{1}$, $\boxed{2}$, $\boxed{1}$, $\boxed{0}$, and $\boxed{0}$.

(b) The winding numbers of γ_2 around -1, 1, $1+i$, $1-i$, and $-1+i$.

- As in (a), by deforming the contour or drawing a ray, we see the desired winding numbers are $\boxed{1}$, $\boxed{0}$, $\boxed{-1}$, $\boxed{-1}$, and $\boxed{0}$.

(c) The values of $\int_{\gamma_1} \frac{1}{z-1}\,dz$ and $\int_{\gamma_2} \frac{1}{z-1}\,dz$.

- From the definition of the winding number, these are just $2\pi i W_{\gamma_1}(1) = \boxed{4\pi i}$ and $2\pi i W_{\gamma_2}(1) = \boxed{0}$.

(d) The values of $\int_{\gamma_1} \frac{1}{z-(1+i)}\,dz$ and $\int_{\gamma_2} \frac{1}{z-(1+i)}\,dz$.

- From the definition of the winding number, these are just $2\pi i W_{\gamma_1}(1+i) = \boxed{2\pi i}$ and $2\pi i W_{\gamma_2}(1+i) = \boxed{-2\pi i}$.

(e) The values of $\int_{\gamma_1} \frac{e^z}{z-1}\,dz$ and $\int_{\gamma_2} \frac{e^z}{z-1}\,dz$.

- These are both of the form $\int_\gamma \frac{f(z)}{z-z_0}\,dz$ for $f(z) = e^z$ and $z_0 = 1$. By Cauchy's integral formula the values are $2\pi i \cdot W_{\gamma_1}(z_0)\, f(z_0) = \boxed{2\pi i \cdot 2e}$ and $2\pi i \cdot W_{\gamma_2}(z_0)\, f(z_0) = \boxed{0}$.

(f) The value of $\int_{\gamma_2} \frac{2z}{z^2-1}\,dz$.

- By partial fraction decomposition, $\frac{2z}{z^2-1} = \frac{1}{z-1} + \frac{1}{z+1}$. Cauchy's integral formula yields $\int_{\gamma_2} \frac{1}{z-1}\,dz = 0$ and $\int_{\gamma_2} \frac{1}{z+1}\,dz = 2\pi i$, so $\int_{\gamma_2} \frac{2z}{z^2-1}\,dz = \int_{\gamma_2} \frac{1}{z-1}\,dz + \int_{\gamma_2} \frac{1}{z+1}\,dz = \boxed{2\pi i}$.

(g) The value of $\int_{\gamma_2} \frac{2}{z^2-2z+2}\,dz$.

- By partial fraction decomposition, $\frac{2}{z^2-2z+2} = \frac{i}{z-(1-i)} - \frac{i}{z-(1+i)}$. Cauchy's integral formula yields $\int_{\gamma_1} \frac{1}{z-(1+i)}\,dz = -2\pi i$ and $\int_{\gamma_1} \frac{1}{z-(1-i)}\,dz = -2\pi i$, so $\int_{\gamma_1} \frac{2}{z^2-2z+2}\,dz = i\int_{\gamma_1} \frac{1}{z-(1-i)}\,dz - i\int_{\gamma_1} \frac{1}{z-(1+i)}\,dz = i(-2\pi i) - i(-2\pi i) = \boxed{0}$.

2. For each function f on each contour γ, calculate $\int_\gamma f(z)\,dz$:

(a) $f(z) = \dfrac{z^3}{z+1}$ on the counterclockwise boundary of the circle $|z| = 4$.

- This is of the form $\int_\gamma \dfrac{f(z)}{z-z_0}\,dz$ for $f(z) = z^3$ and $z_0 = -1$. By Cauchy's integral formula the value is $2\pi i \cdot W_\gamma(z_0)\, f(z_0) = \boxed{-2\pi i}$.

(b) $f(z) = \dfrac{e^z}{z-2}$ on the counterclockwise boundary of the circle $|z| = 4$.

- This is of the form $\int_\gamma \dfrac{f(z)}{z-z_0}\,dz$ for $f(z) = e^z$ and $z_0 = 2$. By Cauchy's integral formula the value is $2\pi i \cdot W_\gamma(z_0)\, f(z_0) = \boxed{2\pi i \cdot e^2}$.

(c) $f(z) = \dfrac{\sin^2(2z)}{z-1}$ on the counterclockwise boundary of the circle $|z| = 4$.

- This is of the form $\int_\gamma \dfrac{f(z)}{z-z_0}\,dz$ for $f(z) = \sin^2(2z)$ and $z_0 = 1$. By Cauchy's integral formula the value is $2\pi i \cdot W_\gamma(z_0)\, f(z_0) = \boxed{2\pi i \cdot \sin^2(2)}$.

(d) $f(z) = \dfrac{z^2+1}{z^2-1}$ on the counterclockwise boundary of the rectangle with vertices ± 20, $\pm 25i$.

- First by partial fraction decomposition we have $f(z) = 1 + \dfrac{1}{z-1} - \dfrac{1}{z+1}$. Note that both 1 and -1 are in γ.
- Then $\int_\gamma 1\,dz = 0$, $\int_\gamma \dfrac{1}{z-1}\,dz = 2\pi i$, and $\int_\gamma \dfrac{1}{z+1}\,dz = 2\pi i$ all by Cauchy's integral formula. So the integral is $\boxed{0}$.

(e) $f(z) = \dfrac{z^2+1}{z^2-1}$ on the counterclockwise boundary of the square with vertices 0, $10-10i$, 20, $10+10i$.

- As in (c), $f(z) = 1 + \dfrac{1}{z-1} - \dfrac{1}{z+1}$. Note that 1 is in γ but -1 is not.
- Then $\int_\gamma 1\,dz = 0$, $\int_\gamma \dfrac{1}{z-1}\,dz = 2\pi i$, and $\int_\gamma \dfrac{1}{z+1}\,dz = 0$ all by Cauchy's integral formula. So the integral is $\boxed{2\pi i}$.

(f) $f(z) = \dfrac{e^z}{z\sin z}$ on the counterclockwise boundary of the circle $|z-\pi| = e^{-\pi}$.

- The function only fails to be holomorphic for $z = 0$ and for $\sin z = 0$ (namely $z = k\pi$ for integers k).
- Since the circle has radius $e^{-\pi} < 1$ we may shrink the circle arbitrarily close to π without changing the value of the integral.
- Then, as a Laurent series centered at $z = \pi$, we can crunch out $\dfrac{e^z}{z\sin z} = -\dfrac{e^\pi}{\pi}(z-\pi)^{-1} - \dfrac{e^\pi}{\pi^2}(\pi - 1) + \cdots$, which converges on a disc of positive radius since the series for e^z and $z\sin z$ converge for all z.
- So by our results on integrating power series we see $\int_\gamma \dfrac{e^z}{z\sin z}\,dz = 2\pi i \cdot (-\dfrac{e^\pi}{\pi}) = \boxed{-2e^\pi i}$.

(g) $f(z) = \overline{z}e^{1/\overline{z}}$ on the counterclockwise boundary of the circle $|z| = 2$. [Hint: On γ, $\overline{z}$ can be written in terms of z.]

- Per the hint we observe that for $|z| = 2$ we have $z\overline{z} = |z|^2 = 4$ so $\overline{z} = 4/z$.
- So on γ we have $f(z) = \dfrac{4}{z}e^{z/4}$, so the integral is $\int_\gamma \dfrac{4e^{z/4}}{z}\,dz$ which by Cauchy's integral formula is $2\pi i \cdot 1 \cdot 4 = \boxed{8\pi i}$.

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3. If γ is the unit circle traversed once counterclockwise, let $I(a,b) = \int_\gamma \frac{1}{(z-a)(z-b)}\,dz$.

   (a) Find $I(a,a)$ if $|a| \neq 1$.

   - Note that $\frac{1}{(z-a)^2}$ has an antiderivative $-\frac{1}{z-a}$, so the integral is $\boxed{0}$ by the fundamental theorem of line integrals.

   (b) Find $I(a,b)$ if $|a| < 1$ and $|b| < 1$ and $a \neq b$.

   - By partial fraction decomposition we have $\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left[\frac{1}{z-a} - \frac{1}{z-b}\right]$. Note also that $\int_\gamma \frac{1}{z-c}\,dz = \begin{cases} 2\pi i & \text{if } |c| < 1 \\ 0 & \text{if } |c| > 1 \end{cases}$.
   - Then $I(a,b) = \frac{1}{a-b}\left[\int_\gamma \frac{1}{z-a}\,dz - \int_\gamma \frac{1}{z-b}\,dz\right] = \frac{1}{a-b}[2\pi i - 2\pi i] = \boxed{0}$.

   (c) Find $I(a,b)$ if $|a| < 1$ and $|b| > 1$.

   - As in (b) we have $I(a,b) = \frac{1}{a-b}\left[\int_\gamma \frac{1}{z-a}\,dz - \int_\gamma \frac{1}{z-b}\,dz\right] = \frac{1}{a-b}[2\pi i] = \boxed{\frac{2\pi i}{a-b}}$.

   (d) Find $I(a,b)$ if $|a| > 1$ and $|b| > 1$ and $a \neq b$.

   - As in (b) we have $I(a,b) = \frac{1}{a-b}\left[\int_\gamma \frac{1}{z-a}\,dz - \int_\gamma \frac{1}{z-b}\,dz\right] = \frac{1}{a-b}[0 - 0] = \boxed{0}$.

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4. The goal of this problem is to give another way to evaluate the integral $I_\gamma(z_0) = \int_\gamma \frac{1}{z-z_0}\,dz$ where γ is any counterclockwise-oriented circle not containing z_0, which was the subject of problem 5 of homework 6.

   (a) Suppose z_0 is a distance $R > 0$ away from the closest point on the circle. Show that $\left|\int_\gamma \frac{1}{z-z_0}\,dz\right| \leq \frac{2\pi r}{R}$ where r is the radius of the circle. [Hint: Bound the integral by the arclength times the maximum of the function.]

   - By problem 7c of homework 6, we have $\left|\int_\gamma \frac{1}{z-z_0}\,dz\right| \leq sM$ where $s = 2\pi r$ is the arclength of the circle and M is the maximum value of $\left|\frac{1}{z-z_0}\right|$ on the circle. But this last quantity is the reciprocal of the minimum distance from z_0 to a point on the circle, so $M = \frac{1}{R}$.
   - Thus we get $\left|\int_\gamma \frac{1}{z-z_0}\,dz\right| \leq sM = \frac{2\pi r}{R}$ as claimed.

   (b) Suppose z_0 is outside the circle. Show that $\int_\gamma \frac{1}{z-z_0}\,dz = 0$. [Hint: Move γ far away from z_0.]

   - Since $f(z) = \frac{1}{z-z_0}$ is holomorphic for $z \neq z_0$, we can move the circle arbitrarily as long it does not cross z_0. But since z_0 is outside the circle, we can move the circle as far away as we like, say a distance $R = \frac{2\pi r}{\epsilon}$ for any $\epsilon > 0$.
   - Then (a) yields $\left|\int_\gamma \frac{1}{z-z_0}\,dz\right| \leq \frac{2\pi r}{R} = \epsilon$. Thus since $\int_\gamma \frac{1}{z-z_0}\,dz$ has absolute value less than ϵ for every $\epsilon > 0$, the integral must be zero.

   (c) Suppose z_0 is in the interior of the circle. Show that $\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i$. [Hint: Recenter γ at z_0.]

   - Since $f(z) = \frac{1}{z-z_0}$ is holomorphic for $z \neq z_0$, we can move the circle arbitrarily as long it does not cross z_0. So we may recenter the circle at z_0.
   - Now just parametrize: if γ is $|z - z_0| = r$ traversed once counterclockwise, taking $\gamma(t) = a + re^{it}$ for $0 \leq t \leq 2\pi$ yields $\int_\gamma \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$ as claimed.

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5. Suppose $p(z)$ is a polynomial of degree at least 2. Then $p(z)$ has finitely many complex zeroes, so they are all contained in a disc $|z| < r$ for some r. For $R \geq r$, let $I(R) = \int_{\gamma_R} \frac{1}{p(z)}\,dz$ where γ_R is the circle $|z| = R$ traversed once counterclockwise.

(a) Show that $I(R) = I(r)$ for all $R \geq r$.

- Note that $\frac{1}{p(z)}$ is holomorphic everywhere except at the zeroes of $p(z)$, which are all contained inside the disc $|z| < r$. So the function is holomorphic everywhere outside $|z| < r$.
- This means we may deform the contour $|z| = r$ into $|z| = R$ without passing through any points where the function is not holomorphic. Hence by deformation of contours, we have $I(r) = I(R)$.

(b) Show that $|I(R)| \leq \frac{2\pi R}{|a|\,(R-r)^d}$ where p has degree d and leading coefficient a. [Hint: You may assume $p(z)$ factors as $p(z) = a(z-z_1)(z-z_2)\cdots(z-z_d)$.]

- By the fundamental theorem of algebra we may suppose $p(z)$ factors as $p(z) = a(z-z_1)(z-z_2)\cdots(z-z_d)$ where by hypothesis $|z_i| < r$ for each i.
- Then if $|z| = R$ we have $|p(z)| = |a|\,|z-z_1|\,|z-z_2|\cdots|z-z_d| \geq |a|\,(|z|-|z_1|)\cdots(|z|-|z_d|) = |a|\,(R-r)^d$ by the triangle inequality applied to each factor.
- This means $\left|\frac{1}{p(z)}\right| \leq \frac{1}{|a|\,(R-r)^d}$ on the contour γ_R. Applying the basic estimate for a contour integral of arclength times maximum, we see $|I(R)| \leq 2\pi R \cdot \frac{1}{|a|\,(R-r)^d}$ as claimed.

(c) Show that $\lim_{R\to\infty} I(R) = 0$. Deduce that $I(r) = 0$.

- Since $d \geq 2$ and $|a|$ and r are fixed constants, we see that $\frac{2\pi R}{|a|\,(R-r)^d} \sim \frac{2\pi}{|a|} R^{1-d} \to 0$ as $R \to 0$. Therefore since $|I(R)| \leq \frac{2\pi R}{|a|\,(R-r)^d}$ by (b) and the upper bound goes to zero as $R \to \infty$, we have $\lim_{R\to\infty} I(R) = 0$.
- Then by (a), since $I(r) = I(R)$ for all $R \geq r$, we have $I(r) = \lim_{R\to\infty} I(r) = \lim_{R\to\infty} I(R) = 0$, as desired.

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6. The goal of this problem is to give a third approach for evaluating the integral $I_\gamma(z_0) = \int_\gamma \frac{1}{z-z_0}\,dz$ where γ is any counterclockwise-oriented circle not containing z_0.

(a) Suppose z_0 is outside the circle. Show that $\int_\gamma \frac{1}{z-z_0}\,dz = 0$. [Hint: Pick a branch of $\log(z-z_0)$ that does not intersect the circle.]

- Since z_0 is outside the circle, we can choose a branch $F(z)$ of $\log(z-z_0)$ that does not intersect the circle (e.g., by taking the branch cut along the ray from z_0 to ∞ in the direction opposite from the center of the circle).
- Then on γ the integrand has an antiderivative $F(z)$, so since γ is closed, by the fundamental theorem of calculus / independence of path, the integral is zero.

(b) Suppose z_0 is in the interior of the circle and let the horizontal ray $z = z_0 + t$ for $t \geq 0$ intersect the circle at P. Choose any α and β on the circle such that the points P, α, β are in counterclockwise order around the circle, and take $\tilde{\gamma}$ to be the counterclockwise arc from α to β. Show that $\int_{\tilde{\gamma}} \frac{1}{z-z_0}\,dz = \mathrm{Log}(\beta-z_0) - \mathrm{Log}(\alpha-z_0)$.

- Since $\frac{1}{z-z_0}$ has an antiderivative $F(z) = \mathrm{Log}(z-z_0)$ for $z - z_0 \notin [0,\infty)$, which is to say, for z not on the horizontal ray $z = z_0 + t$ for $t \geq 0$, and the contour $\tilde{\gamma}$ does not intersect the ray, then by the fundamental theorem of line integrals we have $\int_{\tilde{\gamma}} \frac{1}{z-z_0}\,dz = F(\beta) - F(\alpha) = \mathrm{Log}(\beta-z_0) - \mathrm{Log}(\alpha-z_0)$.

(c) Suppose z_0 is in the interior of the circle. Show that $\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i$. [Hint: In (b), let α approach P from above and β approach P from below.]

- Note that γ is the limit of the curve $\tilde{\gamma}$ as α approaches P from above and β approaches P from below.
- So by continuity of the integral, we have $\int_\gamma \frac{1}{z-z_0}\,dz = \lim_{\beta\to P-} F(\beta) - \lim_{\alpha\to P+} F(\alpha) = [\mathrm{Log}(P - z_0) + 2\pi i] - \mathrm{Log}(P - z_0) = 2\pi i$, since the logarithm argument is $2\pi i$ larger from below than from above.

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7. [Challenge] The goal of this problem is to establish some basic facts about the gamma and zeta functions, which are two special functions with broad utility in complex analysis, number theory, statistics, physics, and various other areas. Recall that for a positive real number α and a complex number z, we have $\alpha^z = e^{z \ln \alpha}$.

(a) Suppose n is a positive integer. Show that $\int_0^\infty t^{n-1}e^{-t}\,dt = (n-1)!$.

- Induction on n. For $n = 1$ we have $\int_0^\infty e^{-t}\,dt = -e^{-t}|_{t=0}^\infty = 1 = 0!$ as claimed.
- For the inductive step suppose $\int_0^\infty t^{n-1}e^{-t}\,dt = (n-1)!$. Then integrating by parts yields $\int_0^\infty t^n e^{-t}\,dt = t^n(-e^{-t})|_{t=0}^\infty - \int_0^\infty nt^{n-1}(-e^{-t})\,dt = n\int_0^\infty t^{n-1}e^{-t}\,dt = n\cdot(n-1)! = n!$ as claimed.

(b) Suppose that $\mathrm{Re}(z) > 1$. Show that the integral $\int_0^\infty t^{z-1}e^{-t}\,dt$ converges absolutely.

- Per the hint, first we show that $\int_0^\infty t^{x-1}e^{-t}\,dt$ converges for positive real x.
- By (a) it converges for positive integral values of x, and since the integrand $t^{x-1}e^{-t}$ is an increasing function of x for $t > 1$, this means $\int_1^\infty t^{x-1}e^{-t}\,dt$ converges for all $x > 1$. Then adding the finite value $\int_0^1 t^{x-1}e^{-t}\,dt$ to it shows that $\int_0^\infty t^{x-1}e^{-t}\,dt$ converges also.
- Now for the complex case, with $z = x+iy$ where $x > 0$, for $t > 0$ we have $\left|t^{z-1}e^{-t}\right| = \left|e^{(x+iy-1)\ln t}e^{-t}\right| = e^{(x-1)\ln t}e^{-t} = t^{x-1}e^{-t}$ So $\int_0^\infty \left|t^{z-1}e^{-t}\right|\,dt = \int_0^\infty t^{x-1}e^{-t}\,dt$ which converges as noted above.
- Thus $\int_0^\infty t^{z-1}e^{-t}\,dt$ converges absolutely as desired.

(c) Let R be the region with $\mathrm{Re}(z) > 1$. Show that $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ is holomorphic on R. [Hint: Differentiate under the integral sign.]

- By (b), since the integral $\int_0^\infty t^{z-1}e^{-t}\,dt$ converges absolutely for $\mathrm{Re}(z) > 1$, we may differentiate under the integral sign to obtain $\Gamma'(z) = \int_0^\infty \frac{d}{dz}\left[t^{z-1}e^{-t}\right]\,dt = \int_0^\infty t^{z-1}\ln t\, e^{-t}\,dt$.
- This integral is also convergent since for example $|\ln t| < t$ for large t and $|\ln t| < t^{-\epsilon}$ for small t.
- Thus Γ is holomorphic as claimed.

(d) Suppose that $\mathrm{Re}(z) > 1$. Show that the series $\sum_{n=1}^\infty \frac{1}{n^z}$ converges absolutely.

- For $z = x + iy$ we have $\left|\frac{1}{n^z}\right| = \left|e^{-(x+iy)\ln n}\right| = e^{-x\ln n} = \frac{1}{n^x}$.
- So we need only show that $\sum_{n=1}^\infty \frac{1}{n^x}$ converges for $x > 1$. But this is a standard fact about p-series: the sum is a Riemann sum for the convergent integral $\int_1^\infty \frac{1}{t^x}\,dt = \frac{t^{1-x}}{1-x}|_{t=1}^\infty = -\frac{1}{1-x}$ hence it converges by the integral test.

(e) Let R be the region with $\mathrm{Re}(z) > 1$. Show that $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$ is holomorphic on R.

- By (d) since the series converges absolutely, we may differentiate term by term to see $\zeta'(z) = \sum_{n=1}^\infty \frac{-\ln n}{n^z}$.
- This series is also convergent by the integral test since$|\ln n| < n^\epsilon$ for large n, so $\left|\frac{-\ln n}{n^z}\right| = \frac{\ln n}{n^x} < n^{\epsilon - x}$, whose integral still converges as long as we take ϵ sufficiently small.
