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1. Suppose $\gamma : [a, b] \to \mathbb{C}$ is a contour. The length of the contour γ is defined as $\int_a^b |\gamma'(t)|\,dt$. Note that for $\gamma(t) = x(t) + iy(t)$ this formula reduces to the familiar arclength formula $s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$.

   (a) Find the length of the contour parametrized by $\gamma(t) = 4e^{3it}$, $0 \le t \le \pi$.

   - We have $\gamma'(t) = 12e^{3it}$ so $|\gamma'(t)| = 12$ and the length is $\int_0^\pi 12\,dt = \boxed{12\pi}$.

   (b) Find the length of the contour parametrized by $\gamma(t) = t + i\cosh(t)$, $0 \le t \le 1$.

   - We have $\gamma'(t) = 1 + i\sinh(t)$ so $|\gamma'(t)| = \sqrt{1 + \sinh^2 t} = \cosh t$ so the length is $\int_0^1 \cosh(t)dt = \boxed{\sinh(1)}$.

   (c) Find the length of the contour parametrized by $\gamma(t) = (1 - 2t) + (3 + t)i$, $0 \le t \le 1$.

   - We have $\gamma'(t) = -2 + i$ so $|\gamma'(t)| = \sqrt{5}$ and the length is $\int_0^1 \sqrt{5}\,dt = \boxed{\sqrt{5}}$.

   (d) Find the length of the contour parametrized by $\gamma(t) = (\pi + et) + (\sqrt{2} - \pi t)i$, $0 \le t \le 1$.

   - We have $\gamma'(t) = e - \pi i$ so $|\gamma'(t)| = \sqrt{e^2 + \pi^2}$ and the length is $\int_0^1 \sqrt{e^2 + \pi^2}\,dt = \boxed{\sqrt{e^2 + \pi^2}}$.

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2. For each function f on each contour γ, calculate $\int_\gamma f(z)\,dz$ without[1] using a parametrization of γ:

   (a) $f(z) = z^{-3}$ on the upper half of the unit circle traversed from $z = 1$ to $z = -1$.

   - Since $f(z)$ has an antiderivative $F(z) = -\frac{1}{2}z^{-2}$, by the fundamental theorem of line integrals we have $\int_\gamma f(z)\,dz = F(-1) - F(1) = \boxed{0}$.

   (b) $f(z) = z^n$ (n an integer with $n \neq -1$) on the counterclockwise boundary of the square with vertices ± 1, $\pm i$.

   - Since $f(z)$ has an antiderivative $F(z) = \dfrac{z^{n+1}}{n+1}$ and the contour is closed, by the fundamental theorem of line integrals we have $\int_\gamma f(z)\,dz = \boxed{0}$.

   (c) $f(z) = 3z^2$ on the curve $\gamma(t) = e^{e^t} - \tan(t^2)i$ for $0 \le t \le 1$.

   - Since $f(z)$ has an antiderivative $F(z) = z^3$, by the fundamental theorem of line integrals we have $\int_\gamma f(z)\,dz = F(\gamma(1)) - F(\gamma(0)) = \boxed{(e - i\tan 1)^3 - e^3}$.

   (d) $f(z) = e^z$ on the portion of the ellipse $\dfrac{\mathrm{Re}(z)^2}{(\ln 2)^2} + \dfrac{\mathrm{Im}(z)^2}{4\pi^2} = 1$ clockwise from $z = \ln 2$ to $z = 2\pi i$.

   - Since $f(z)$ has an antiderivative $F(z) = e^z$, by the fundamental theorem of line integrals we have $\int_\gamma f(z)\,dz = F(2\pi i) - F(\ln 2) = \boxed{e^{2\pi i} - e^{\ln 2}} = \boxed{-1}$.

   (e) $f(z) = 1/z$ on the polygonal path with successive vertices i, $-1 + 7i$, $-20 - 25i$, and $-i$.

   - Note $1/z$ has an antiderivative $F(z) = \mathrm{Log}(z)$ on $\mathbb{C}\backslash[0, \infty)$ inside which this path lies.
   - So by the fundamental theorem of line integrals, $\int_\gamma f(z)\,dz = F(-i) - F(i) = \mathrm{Log}(-i) - \mathrm{Log}(i) = \boxed{i\pi}$.

   (f) $f(z) = \mathrm{Log}(z)$ on the polygonal path with successive vertices i, $-1 + 7i$, $-20 - 25i$, and $-i$.

   - Note that $\mathrm{Log}(z)$ is holomorphic on $\mathbb{C}\backslash[0, \infty)$ inside which this path lies.
   - By the product rule we see $F(z) = z\,\mathrm{Log}(z) - z$ has derivative $f(z) = \mathrm{Log}(z)$, so by the fundamental theorem of line integrals, $\int_\gamma f(z)\,dz = F(-i) - F(i) = [(-i)\mathrm{Log}(-i) - (-i)] - [i\,\mathrm{Log}(i) - i] = \boxed{2\pi + 2i}$.

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[1] Obviously, since this is in Part I, we won't actually know for sure that you didn't use a parametrization, but read between the lines on the directions here (namely, that you should be able to solve each part without using a parametrization).

3. For each function f on each closed contour γ, calculate $\int_\gamma f(z)\,dz$:

   (a) $f(z) = z^{-1}$ on the counterclockwise boundary of the triangle with vertices 1 and $-2 \pm i\sqrt{3}$.

   - Note f is holomorphic for all $z \neq 0$.
   - We can deform this contour in $\mathbb{C}\backslash\{0\}$ into the unit circle, and thus the integral is $\boxed{2\pi i}$.

   (b) $f(z) = z^{-1}$ on the counterclockwise boundary of the triangle with vertices -1 and $-2 \pm i\sqrt{3}$.

   - Note f is holomorphic for all $z \neq 0$.
   - Since this triangle does not contain $z = 0$, we can deform this contour to a point, and so the integral is $\boxed{0}$.

   (c) $f(z) = z^{-1}$ on the polygonal path with successive vertices 1, i, -1, $-i$, 2, $2i$, -2, $-2i$, and 1.

   - Note f is holomorphic for all $z \neq 0$.
   - We can deform this contour in $\mathbb{C}\backslash\{0\}$ into a path that winds twice around the unit circle, so the integral is $\boxed{4\pi i}$.

   (d) $f(z) = \dfrac{i}{z^2 - iz}$ on the counterclockwise boundary of the ellipse $\dfrac{\text{Re}(z)^2}{(\ln 2)^2} + \dfrac{\text{Im}(z)^2}{4\pi^2} = 1$.

   - Note f is holomorphic for all $z \neq 0, i$, and both of these points lie in the ellipse.
   - Furthermore, from partial fraction decomposition we have $\dfrac{i}{z^2 - iz} = \dfrac{1}{z - i} - \dfrac{1}{z}$, so we need only calculate $\int_\gamma \dfrac{1}{z}\,dz$ and $\int_\gamma \dfrac{1}{z - i}\,dz$.
   - For $\int_\gamma \dfrac{1}{z}\,dz$ we deform the contour into the unit circle to see the integral is $2\pi i$. For $\int_\gamma \dfrac{1}{z - i}\,dz$ we deform the contour into the circle $|z - i| = 1$ and then substitute $w = z - i$ to see that the integral is $2\pi i$.
   - Therefore we have $\int_\gamma \dfrac{i}{z^2 - iz}\,dz = \int_\gamma \dfrac{1}{z - i}\,dz - \int_\gamma \dfrac{1}{z}\,dz = \boxed{0}$.

   (e) $f(z) = \dfrac{z}{z^2 + 1}$ on the counterclockwise boundary of the square with vertices ± 1 and $\pm 1 + 2i$.

   - Note f is holomorphic for all $z \neq i, -i$, and i lies inside the square while $-i$ does not.
   - Furthermore, from partial fraction decomposition we have $\dfrac{z}{z^2 + 1} = \dfrac{1/2}{z + i} + \dfrac{1/2}{z - i}$, so we need only calculate $\int_\gamma \dfrac{1}{z + i}\,dz$ and $\int_\gamma \dfrac{1}{z - i}\,dz$.
   - Since $-i$ is not in the square we see that $\int_\gamma \dfrac{1}{z + i}\,dz$ by deforming the contour to a point. For $\int_\gamma \dfrac{1}{z - i}\,dz$ we deform the contour to the circle $|z - i| = 1$ to see that the integral is $2\pi i$ as earlier.
   - Therefore we have $\int_\gamma \dfrac{1}{z^2 + 1}\,dz = \dfrac{1}{2}\int_\gamma \dfrac{1}{z + i}\,dz + \dfrac{1}{2}\int_\gamma \dfrac{1}{z - i}\,dz = \boxed{\pi i}$.

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4. Let $f(z)$ be continuous on a connected open region R. Prove that the following are equivalent (you may freely appeal to theorems proven in class or in the notes; if you find yourself writing a long solution then you've missed something):

   (a) The line integral $\int_\gamma f(z)\,dz = 0$ for all closed contours γ in R.

   (b) There exists a holomorphic $F(z)$ on R such that $F'(z) = f(z)$.

   (c) For any $a, b \in R$ and any contours γ_1 and γ_2 in R from a to b, $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.

   - (a) implies (b): This is the existence of antiderivatives theorem from the notes.
   - (b) implies (c): This is independence of path: $\int_{\gamma_1} f(z)\,dz = F(b) - F(a) = \int_{\gamma_2} f(z)\,dz$.
   - (c) implies (a): Suppose γ is closed. Take $\gamma_1 = \gamma$ and γ_2 to be constant at any point on γ. Then $\int_{\gamma_2} f(z)\,dz = 0$, so the hypothesis of (c) yields immediately $\int_\gamma f(z)\,dz = 0$.

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5. The goal of this problem is to compute the integral $I(z_0) = \int_\gamma \frac{dz}{z - z_0}$ where γ is the unit circle traversed once counterclockwise and $|z_0| \neq 1$.

   (a) If $z_0 = re^{i\theta}$ show that $I(z_0) = I(r)$. [Hint: Make a substitution.]

   - We substitute $z = e^{i\theta}\tilde{z}$, with $dz = e^{i\theta} d\tilde{z}$. This substitution corresponds to a rotation of θ radians (clockwise going from z to $\tilde{z}$, or counterclockwise going from $\tilde{z}$ to z), which does not change the contour γ.
   - Then we have $I(z_0) = \int_\gamma \frac{dz}{z - z_0} = \int_\gamma \frac{e^{i\theta} d\tilde{z}}{e^{i\theta}\tilde{z} - re^{i\theta}} = \int_\gamma \frac{d\tilde{z}}{\tilde{z} - r} = I(r)$ as required.

   (b) Suppose $r \geq 0$ and $r \neq 1$. Show that $\int_0^{2\pi} \frac{r \sin t}{1 - 2r\cos t + r^2}\, dt = 0$.

   - Observe that the derivative of $\ln(1 - 2r\cos t + r^2)$ is $\frac{2r \sin t}{1 - 2r\cos t + r^2}$. Since the argument of the logarithm ranges from $(r-1)^2$ to $(r+1)^2$ and $r \neq 1$, it is never zero.
   - Thus by the fundamental theorem of calculus we have $\int_0^{2\pi} \frac{r \sin t}{1 - 2r\cos t + r^2}\, dt = \frac{1}{2}\ln(1 - 2r\cos t + r^2)\big|_{t=0}^{2\pi} = 0$ by periodicity.

   (c) Show that $\int_0^{\pi} \frac{1 - r\cos t}{1 - 2r\cos t + r^2}\, dt = \begin{cases} \pi & \text{if } 0 \leq r < 1 \\ 0 & \text{if } r > 1 \end{cases}$. [Hint: Substitute $x = \tan(t/2)$, which has $dt = \frac{2dx}{1+x^2}$ and $\cos t = \frac{1-x^2}{1+x^2}$. You may want to make a computer do the partial fraction decomposition.]

   - Making the suggested substitution $x = \tan(t/2)$, which has $dt = \frac{2dx}{1+x^2}$ and $\cos t = \frac{1-x^2}{1+x^2}$ and has an integration range of $(-\infty, \infty)$ for x yields

$$\begin{aligned}
\int_0^{\pi} \frac{1 - r\cos t}{1 - 2r\cos t + r^2}\, dt &= 2\int_0^{\infty} \frac{1 - r\cdot\frac{1-x^2}{1+x^2}}{1 - 2r\cdot\frac{1-x^2}{1+x^2} + r^2} \cdot \frac{dx}{1+x^2} \\
&= 2\int_0^{\infty} \frac{(1+x^2) - r(1-x^2)}{(1+r^2)(1+x^2) - 2r(1-x^2)} \cdot \frac{dx}{1+x^2} \\
&= 2\int_0^{\infty} \frac{(1-r) + (1+r)x^2}{(1-2r+r^2) + (1+2r+r^2)x^2} \cdot \frac{dx}{1+x^2} \\
&= 2\int_0^{\infty} \left[\frac{1/2}{1+x^2} - \frac{(r^2-1)/2}{(1-2r+r^2) + (1+2r+r^2)x^2}\right] dx \\
&= \int_0^{\infty} \frac{1}{1+x^2}\, dx - \int_0^{\infty} \frac{1}{1 + [\frac{r+1}{r-1}x]^2} \frac{r+1}{r-1} dx
\end{aligned}$$

   - The first integral is $\frac{\pi}{2}$ while substituting $u = \frac{r+1}{r-1}x$ in the second integral yields $\int_0^\infty \frac{du}{1+u^2} = \frac{\pi}{2}$ also, but with the order of integration preserved for $r > 1$ and reversed for $r < 1$. So for $r > 1$ the two integrals cancel and the sum is zero, while for $r < 1$ the integrals add and the sum is π.

   (d) Find the value of $I(z_0)$ in terms of z_0.

   - By (a) we just need to evaluate $I(r)$. With $\gamma(t) = e^{it}$ for $0 \leq t \leq 2\pi$ we have $\gamma'(t) = ie^{it}$ and $\frac{1}{z - r} = \frac{1}{e^{it} - r}$.
   - Plugging in the parametrization then yields $\int_\gamma \frac{1}{z - r}\, dz = \int_0^{2\pi} \frac{1}{e^{it} - r} \cdot ie^{it}\, dt = i\int_0^{2\pi} \frac{\cos t + i\sin t}{(\cos t - r) + i\sin t}\, dt$ $= i\int_0^{2\pi} \frac{(\cos t + i\sin t)(\cos t - r - i\sin t)}{(\cos t - r)^2 + \sin^2 t}\, dt = i\int_0^{2\pi} \frac{1 - r\cos t - ri\sin t}{1 - 2r\cos t + r^2}\, dt$.
   - By (b) and (c) we see that this integral equals 0 when $r > 1$ and $i \cdot 2\pi$ when $r < 1$. So $I(z_0) = \boxed{2\pi i \text{ for } |z_0| < 1 \text{ and } 0 \text{ for } |z_0| > 1}$.

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6. Recall that if $\gamma_0 : [0,1] \to \mathbb{C}$ and $\gamma_1 : [0,1] \to \mathbb{C}$ are continuous closed curves in a region R, we say they are homotopic in R if there exists some continuous function $h : [0,1] \times [0,1] \to R$ with $h(s,0) = h(s,1)$ for all s and $h(0,t) = \gamma_0(t)$ and $h(1,t) = \gamma_1(t)$. Prove that being homotopic in R is an equivalence relation on continuous closed curves in R. [Hint: For transitivity, use half of the interval to go from γ_0 to γ_1 and the other half to go from γ_1 to γ_2.]

   - Reflexive: To show γ is homotopic to itself, simply take the trivial homotopy, with $h(s,t) = \gamma(t)$ for all s.
   - Symmetric: If γ_0 is homotopic to γ_1 with $h(s,0) = h(s,1)$, $h(0,t) = \gamma_0(t)$, and $h(1,t) = \gamma_1(t)$, then let $\tilde{h}(s,t) = h(1-s,t)$. Then $\tilde{h}$ has $\tilde{h}(s,0) = h(s,1) = h(s,0) = \tilde{h}(s,1)$ and $\tilde{h}(0,t) = h(1,t) = \gamma_1(t)$ and $\tilde{h}(1,t) = h(0,t) = \gamma_0(t)$ so $\tilde{h}$ is a homotopy from γ_1 to γ_0.
   - Transitive: Suppose γ_0 is homotopic to γ_1 with $h_1(s,0) = h_1(s,1)$, $h_1(0,t) = \gamma_0(t)$, and $h_1(1,t) = \gamma_1(t)$, and also that γ_1 is homotopic to γ_2 with $h_1(s,0) = h_1(s,1)$, $h_1(0,t) = \gamma_1(t)$, and $h_1(1,t) = \gamma_2(t)$. Then let $h(s,t) = \begin{cases} h_1(2s,t) & \text{for } 0 \le s \le 1/2 \\ h_2(2s-1,t) & \text{for } 1/2 < s \le 1 \end{cases}$.
   - Since h_1 and h_2 are continuous and agree on the boundary $s = 1/2$, since $h_1(1,t) = \gamma_1(t) = h_2(0,t)$ for each t, we see h is continuous. Furthermore $h(s,0) = \begin{cases} h_1(s,0) & \text{for } 0 \le s \le 1/2 \\ h_2(s,0) & \text{for } 1/2 < s \le 1 \end{cases} = \begin{cases} h_1(s,1) & \text{for } 0 \le s \le 1/2 \\ h_2(s,1) & \text{for } 1/2 < s \le 1 \end{cases} = h(s,1)$, and $h(0,t) = \gamma_0(t)$ and $h(1,t) = \gamma_2(t)$. So h is a homotopy from γ_0 to γ_2 as desired.

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7. [Challenge] Recall the definition of the length of a contour from problem 1.

   (a) Suppose that $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ are continuously differentiable and there exists a continuously differentiable increasing function $g : [a,b] \to [c,d]$ with $g(a) = c$ and $g(b) = d$ such that $\gamma_1 = \gamma_2 \circ g$, show that $\int_a^b |\gamma_1'(t)|\, dt = \int_c^d |\gamma_2'(s)|\, ds$. [Hint: Substitution.]

   - Make the substitution $t = g(s)$. For $t = a$ we have $s = g(a) = c$ and for $t = b$ we have $s = g(b) = d$, and also $dt = g'(s)\, ds$. Additionally note that $g'(s) > 0$ by the assumption that g is increasing.
   - Then substituting $t = g(s)$ in $\int_a^b |\gamma_1'(t)|\, dt$ yields $\int_c^d |\gamma_1'(g(s))|\, g'(s)\, ds = \int_c^d |\gamma_1'(g(s))g'(s)|\, ds = \int_c^d |\gamma_2'(s)|\, ds$ by the chain rule and the fact that $g'(s) = |g'(s)|$ since $g'(s) > 0$.

   (b) Prove that the length of a contour is independent of the parametrization. [Hint: Sum (a).]

   - Suppose we have two different parametrizations of the same contour. By breaking each at the points where either one is nondifferentiable, we may assume that $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$ and $\tilde{\gamma} = \tilde{\gamma}_1 \cup \cdots \cup \tilde{\gamma}_n$ where each γ_i and $\tilde{\gamma}_i$ is continuously differentiable and has the same endpoints.
   - Then applying (a) to each pair $(\gamma_i, \tilde{\gamma}_i)$ we see that the length of γ_i equals the length of $\tilde{\gamma}_i$. Summing over all pieces and observing that the length integral is additive shows immediately that the length of γ equals the length of $\tilde{\gamma}$.

   (c) Suppose that $|f(z)| \le M$ on the contour γ of length s. Prove that $\left| \int_\gamma f(z)\, dz \right| \le Ms$.

   - Suppose $\gamma : [a,b] \to \mathbb{C}$ is a contour. Then $\left| \int_\gamma f(z)\, dz \right| = \left| \int_a^b f(\gamma(t))\, \gamma'(t)\, dt \right| \le \int_a^b |f(\gamma(t))|\, |\gamma'(t)|\, dt \le \int_a^b M\, |\gamma'(t)|\, dt = M \int_a^b |\gamma'(t)|\, dt = Ms$ using the triangle inequality and the fact that $|f(\gamma(t))| \le M$ for all t.
