E. Dummit's Math 4555 ~ Complex Analysis, Fall 2025 ~ Homework 6, due Fri Oct 17th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Suppose $\gamma : [a,b] \to \mathbb{C}$ is a contour. The <u>length</u> of the contour γ is defined as $\int_a^b |\gamma'(t)|\, dt$. Note that for $\gamma(t) = x(t) + iy(t)$ this formula reduces to the familiar arclength formula $s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$.
   (a) Find the length of the contour parametrized by $\gamma(t) = 4e^{3it}$, $0 \le t \le \pi$.
   (b) Find the length of the contour parametrized by $\gamma(t) = t + i\cosh(t)$, $0 \le t \le 1$.
   (c) Find the length of the contour parametrized by $\gamma(t) = (1-2t) + (3+t)i$, $0 \le t \le 1$.
   (d) Find the length of the contour parametrized by $\gamma(t) = (\pi + et) + (\sqrt{2} - \pi t)i$, $0 \le t \le 1$.

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2. For each function f on each contour γ, calculate $\int_\gamma f(z)\, dz$ without[1] using a parametrization of γ:
   (a) $f(z) = z^{-3}$ on the upper half of the unit circle traversed from $z = 1$ to $z = -1$.
   (b) $f(z) = z^n$ (n an integer with $n \ne -1$) on the counterclockwise boundary of the square with vertices ± 1, $\pm i$.
   (c) $f(z) = 3z^2$ on the curve $\gamma(t) = e^{e^t} - \tan(t^2)i$ for $0 \le t \le 1$.
   (d) $f(z) = e^z$ on the portion of the ellipse $\dfrac{\mathrm{Re}(z)^2}{(\ln 2)^2} + \dfrac{\mathrm{Im}(z)^2}{4\pi^2} = 1$ clockwise from $z = \ln 2$ to $z = 2\pi i$.
   (e) $f(z) = 1/z$ on the polygonal path with successive vertices i, $-1 + 7i$, $-20 - 25i$, and $-i$.
   (f) $f(z) = \mathrm{Log}(z)$ on the polygonal path with successive vertices i, $-1 + 7i$, $-20 - 25i$, and $-i$.

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3. For each function f on each closed contour γ, calculate $\int_\gamma f(z)\, dz$:
   (a) $f(z) = z^{-1}$ on the counterclockwise boundary of the square with vertices 1 and $-2 \pm i\sqrt{3}$.
   (b) $f(z) = z^{-1}$ on the counterclockwise boundary of the triangle with vertices -1 and $-2 \pm i\sqrt{3}$.
   (c) $f(z) = z^{-1}$ on the polygonal path with successive vertices 1, i, -1, $-i$, 2, $2i$, -2, $-2i$, and 1.
   (d) $f(z) = \dfrac{i}{z^2 - iz}$ on the counterclockwise boundary of the ellipse $\dfrac{\mathrm{Re}(z)^2}{(\ln 2)^2} + \dfrac{\mathrm{Im}(z)^2}{4\pi^2} = 1$.
   (e) $f(z) = \dfrac{z}{z^2 + 1}$ on the counterclockwise boundary of the square with vertices ± 1 and $\pm 1 + 2i$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Let $f(z)$ be continuous on a connected open region R. Prove that the following are equivalent (you may freely appeal to theorems proven in class or in the notes; if you find yourself writing a long solution then you've missed something):
   (a) The line integral $\int_\gamma f(z)\, dz = 0$ for all closed contours γ in R.
   (b) There exists a holomorphic $F(z)$ on R such that $F'(z) = f(z)$.
   (c) For any $a, b \in R$ and any contours γ_1 and γ_2 in R from a to b, $\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$.

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[1] Obviously, since this is in Part I, we won't actually know for sure that you didn't use a parametrization, but read between the lines on the directions here (namely, that you should be able to solve each part without using a parametrization).

5. The goal of this problem is to compute the integral $I(z_0) = \int_\gamma \frac{dz}{z - z_0}$ where γ is the unit circle traversed once counterclockwise and $|z_0| \neq 1$.

   (a) If $z_0 = re^{i\theta}$ show that $I(z_0) = I(r)$. [Hint: Make a substitution.]

   (b) Suppose $r \geq 0$ and $r \neq 1$. Show that $\int_0^{2\pi} \frac{r \sin t}{1 - 2r\cos t + r^2}\, dt = 0$.

   (c) Show that $\int_0^{\pi} \frac{1 - r\cos t}{1 - 2r\cos t + r^2}\, dt = \begin{cases} \pi & \text{if } 0 \leq r < 1 \\ 0 & \text{if } r > 1 \end{cases}$. [Hint: Substitute $x = \tan(t/2)$, which has $dt = \frac{2dx}{1+x^2}$ and $\cos t = \frac{1-x^2}{1+x^2}$. You may want to make a computer do the partial fraction decomposition.]

   (d) Find the value of $I(z_0)$ in terms of z_0.

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6. Recall that if $\gamma_0 : [0,1] \to \mathbb{C}$ and $\gamma_1 : [0,1] \to \mathbb{C}$ are continuous curves in a region R, we say they are <u>homotopic</u> in R if there exists some continuous function $h : [0,1] \times [0,1] \to R$ with $h(s,0) = h(s,1)$ for all s and $h(0,t) = \gamma_0(t)$ and $h(1,t) = \gamma_1(t)$. Prove that being homotopic in R is an equivalence relation on continuous curves in R. [Hint: For transitivity, use half of the interval to go from γ_0 to γ_1 and the other half to go from γ_1 to γ_2.]

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7. [Challenge] Recall the definition of the length of a contour from problem 1.

   (a) Suppose that $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ are continuously differentiable and there exists a continuously differentiable increasing function $g : [a,b] \to [c,d]$ with $g(a) = c$ and $g(b) = d$ such that $\gamma_1 = \gamma_2 \circ g$, show that $\int_a^b |\gamma_1'(t)|\, dt = \int_c^d |\gamma_2'(s)|\, ds$. [Hint: Substitution.]

   (b) Prove that the length of a contour is independent of the parametrization. [Hint: Sum (a).]

   (c) Suppose that $|f(z)| \leq M$ on the contour γ of length s. Prove that $\left|\int_\gamma f(z)\, dz\right| \leq Ms$.
