1. Find all possible complex values for each of these expressions (note that log always denotes the multivalued complex logarithm):

   (a) $\log(e)$.

   - Since $e = e^1$ we have $\log(e) = \boxed{1 + 2k\pi i}$ for $k \in \mathbb{Z}$.

   (b) $\log(i)$.

   - Since $i = e^{i\pi/2}$ we have $\log(i) = \boxed{(\pi/2 + 2k\pi)i}$ for $k \in \mathbb{Z}$.

   (c) i^{2i}.

   - By definition we have $a^b = e^{b\log(a)}$. Since $\log(i) = (\pi/2+2k\pi)i$ from (a), we see $i^i = e^{2i[(\pi/2+2k\pi)i]} = \boxed{e^{-\pi-4k\pi}}$ for $k \in \mathbb{Z}$.

   (d) 4^i.

   - By definition we have $a^b = e^{b\log(a)}$, so since $\log 4 = \ln 4 + 2k\pi i$, we have $4^i = e^{i[\ln(4)+2k\pi i]} = e^{i\ln 4 - 2k\pi} = \boxed{e^{-2k\pi}[\cos(\ln 4) + i\sin(\ln 4)]}$ for $k \in \mathbb{Z}$.

   (e) $e^{\log(i)}$.

   - From (b) we have $\log(i) = (\pi/2 + 2k\pi)i$ so $e^{\log(i)} = e^{i\pi/2}e^{2k\pi i} = \boxed{i}$.

   (f) $\log(e^i)$.

   - Since e^i is already in polar form we have $\log(e^i) = \boxed{i + 2k\pi i}$ for $k \in \mathbb{Z}$.

   (g) $1^{1/6}$.

   - By definition we have $a^b = e^{b\log(a)}$. Since $\log(1) = 2\pi k i$, we have $1^{1/6} = e^{2\pi k i/6}$ for $k \in \mathbb{Z}$.
   - There are six possible values of this expression: $\boxed{e^0, e^{2\pi i/6}, e^{4\pi i/6}, e^{6\pi i/6}, e^{8\pi i/6}, e^{10\pi i/6}}$. We can see that these are just the six different sixth roots of unity.

   (h) $(1^{1/6})^2$.

   - Squaring the six values from (f) yields three values, the three cube roots of unity: $\boxed{e^0, e^{4\pi i/6}, e^{8\pi i/6}}$.

   (i) 1^2.

   - By definition we have $a^b = e^{b\log(a)}$. Since $\log(1) = 2k\pi i$, we see $1^2 = e^{2(2k\pi i)} = e^{4k\pi i} = \boxed{1}$. (So here, in fact the entirely sensible expression 1^2 really does only have one possible complex value!)

   (j) $(1^2)^{1/6}$.

   - Since $1^2 = 1$ this is the same six values as (g): $\boxed{e^0, e^{2\pi i/6}, e^{4\pi i/6}, e^{6\pi i/6}, e^{8\pi i/6}, e^{10\pi i/6}}$. Note in particular that this answer is *not* the same as (h)!

   (k) $1^{2/6}$.

   - By definition we have $a^b = e^{b\log(a)}$. Since $\log(1) = 2\pi k i$, we have $1^{2/6} = 1^{1/3} = e^{2\pi k i/3}$ for $k \in \mathbb{Z}$.
   - There are three possible values of this expression: $\boxed{e^0, e^{2\pi i/3}, e^{4\pi i/3}}$. We can see that these are just the three different cube roots of unity. Note in particular that $1^{2/6}$ is *not* the same as $(1^2)^{1/6}$ from (j)!

   (l) $(-1)^{1/\pi}$.

   - By definition we have $a^b = e^{b\log(a)}$. Since $\log(-1) = (\pi + 2k\pi)i$, we see $(-1)^{1/\pi} = e^{[(\pi+2k\pi)i]/\pi} = e^{(1+2k)i} = \boxed{\cos(2k+1) + i\sin(2k+1)}$ for $k \in \mathbb{Z}$.

2. For each function f on each curve γ, calculate $\int_\gamma f(z)\,dz$:

   (a) $f(z) = 1/z$, $\gamma(t) = e^{it}$ for $0 \le t \le \pi/2$.
   - We compute $\gamma'(t) = ie^{it}$ and $f(z) = z^{-1} = e^{-it}$.
   - Thus $\int_\gamma z^{-1}\,dz = \int_0^\pi e^{-it} \cdot ie^{it}\,dt = \int_0^{\pi/2} i\,dt = \boxed{i\pi/2}$.

   (b) $f(z) = z^n$ (n an integer with $n \ne -1$), $\gamma(t) = e^{it}$ for $0 \le t \le \pi/2$.
   - We compute $\gamma'(t) = ie^{it}$ and $f(z) = z^n = e^{nit}$.
   - Thus $\int_\gamma z^n\,dz = \int_0^{\pi/2} e^{nit} \cdot ie^{it}\,dt = \int_0^{\pi/2} ie^{(n+1)it}\,dt = \dfrac{e^{(n+1)it}}{n+1}\Big|_{t=0}^{\pi/2} = \boxed{\dfrac{e^{(n+1)i\pi/2} - 1}{n+1}}$.

   (c) $f(z) = 3z^2$, $\gamma(t) = t + (1-t)i$ for $0 \le t \le 1$.
   - We compute $\gamma'(t) = 1 - i$ and $f(z) = 3[t + (1-t)i]^2$.
   - Thus $\int_\gamma 3z^2\,dz = \int_0^1 3[t + (1-t)i]^2 \cdot (1-i)\,dt = \int_0^1 [(-3 + 12t - 6t^2) + (3 - 6t^2)i]\,dt = \boxed{1+i}$.
   - Alternatively, since $f(z)$ is holomorphic with antiderivative $F(z) = z^3$, by path independence the value is $F(\gamma(1)) - F(\gamma(0)) = 1 - (-i) = \boxed{1+i}$.

   (d) $f(z) = 3\overline{z}^2$, $\gamma(t) = t + (1-t)i$ for $0 \le t \le 1$.
   - We compute $\gamma'(t) = 1 - i$ and $f(z) = 3[t - (1-t)i]^2$.
   - Thus $\int_\gamma 3\overline{z}^2\,dz = \int_0^1 3[t - (1-t)i]^2) \cdot (1-i)\,dt = \int_0^1 [(-3 + 6t^2) + (3 - 12t + 6t^2)i]\,dt = \boxed{-1-i}$.

   (e) $f(z) = \mathrm{Log}(z)$, $\gamma(t) = 2e^{it}$ for $0 \le t \le 2\pi$, where (as usual) Log denotes the principal complex logarithm.
   - We compute $\gamma'(t) = 2ie^{it}$ and $f(z) = \mathrm{Log}(2e^{it}) = \ln(2) + it$ for $0 \le t < 2\pi$. Since the integral is not affected by the jump discontinuity at $t = 2\pi$, we can ignore it.
   - Then $\int_\gamma \mathrm{Log}(z)\,dz = \int_0^{2\pi} [\ln(2) + it] \cdot 2ie^{it}\,dt = \int_0^{2\pi} [-2te^{it} + 2\ln(2)ie^{it}]\,dt = (-2e^{it} + 2ite^{it} + 2\ln(2)e^{it})|_{t=0}^{2\pi} = \boxed{4\pi i}$.

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3. For each contour γ, give a parametrization and then evaluate $\int_\gamma f(z)\,dz$:

   (a) The circle $|z - 1| = 2$ traversed twice counterclockwise, where $f(z) = \overline{z}$.
   - We can take $\gamma(t) = 1 + 2e^{it}$ for $0 \le t \le 4\pi$.
   - Then $\gamma'(t) = 2ie^{it}$ and $f(z) = \overline{z} = 1 + 2e^{-it}$.
   - Then $\int_\gamma f(z)\,dz = \int_0^{4\pi} (1 + 2e^{-it})(2ie^{it})\,dt = \int_0^{4\pi} [2ie^{it} + 4i]\,dt = \boxed{16\pi i}$.

   (b) The path that starts at 0, follows a straight line to $1 + 2i$, and then follows a straight line to $4 + 2i$, where $f(z) = z$.
   - There are two pieces. For the piece from 0 to $1 + 2i$ we can take $\gamma_1(t) = (1 + 2i)t$ for $0 \le t \le 1$ and for the piece from $1 + 2i$ to $4 + 2i$ we can take $\gamma_2(t) = (1 + 2i) + 3t$ for $0 \le t \le 1$.
   - For γ_1 we get $\gamma_1'(t) = 1 + 2i$ and $f(z) = (1 + 2i)t$ so $\int_{\gamma_1} f(z)\,dz = \int_0^1 (1 + 2i)^2 t\,dt = -3/2 + 2i$.
   - For γ_2 we get $\gamma_2'(t) = 3$ and $f(z) = (1 + 2i) + 3t$ so $\int_{\gamma_2} f(z)\,dz = \int_0^1 3[(1 + 2i) + 3t)]\,dt = 15/2 + 6i$.
   - So the desired integral is the sum $(-3/2 + 2i) + (15/2 + 6i) = \boxed{6 + 8i}$.
   - Alternatively, using the fact that f is holomorphic with antiderivative $F(z) = z^2/2$, by independence of path / the fundamental theorem of calculus for holomorphic functions we see the integral $\int_\gamma f(z)\,dz = F(4 + 2i) - F(0) = \boxed{6 + 8i}$.

   (c) The counterclockwise boundary of the triangle with vertices 0, 2, and $1 + i$, where $f(z) = \mathrm{Re}(z)$.
   - There are three pieces. For the piece from 0 to 2 we can take $\gamma_1(t) = 2t$ for $0 \le t \le 1$, for the piece from 2 to $1 + i$ we can take $\gamma_2(t) = (1 - t)2 + t(1 + i) = (2 - t) + it$ for $0 \le t \le 1$, and for the piece from $1 + i$ to 0 we can take $\gamma_3(t) = (1 - t)(1 + i) + t \cdot 0 = (1 - t) + (1 - t)i$ for $0 \le t \le 1$.
   - For γ_1 we get $\gamma_1'(t) = 2$ and $f(z) = 2t$ so $\int_{\gamma_1} f(z)\,dz = \int_0^1 (2t)(2)\,dt = 2$.

- For γ_2 we get $\gamma_2'(t) = -1+i$ and $f(z) = 2-t$ so $\int_{\gamma_2} f(z)\,dz = \int_0^1 (2-t)(-1+i)\,dt = -\dfrac{3}{2} + \dfrac{3}{2}i$.
- For γ_3 we get $\gamma_3'(t) = -1-i$ and $f(z) = 1-t$ so $\int_{\gamma_3} f(z)\,dz = \int_0^1 (1-t)(-1-i)\,dt = -\dfrac{1}{2} - \dfrac{1}{2}i$.
- So the desired integral is the sum $2 + (-\dfrac{3}{2} + \dfrac{3}{2}i) + (-\dfrac{1}{2} - \dfrac{1}{2}i) = \boxed{i}$.

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4. By differentiating a power series expansion, we can often show it satisfies a differential equation inside its radius of convergence. Show the following (make sure to check the radius of convergence!):

(a) Show that $f(z) = \sum_{n=0}^{\infty} \dfrac{z^{2n}}{(2n)!}$ satisfies $f''(z) = f(z)$ for all $z \in \mathbb{C}$.

- First, by the ratio test, with $a_{2n} = \dfrac{1}{(2n)!}$, we see that $\dfrac{a_{2n+2}}{a_{2n}} = \dfrac{1}{(2n+1)(2n+2)} \to 0$ as $n \to \infty$, so the series has radius of convergence ∞.
- Then differentiating term-by-term yields $f'(z) = \sum_{n=0}^{\infty} \dfrac{2nz^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \dfrac{z^{2n-1}}{(2n-1)!}$ and then $f''(z) = \sum_{n=1}^{\infty} \dfrac{(2n-1)z^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty} \dfrac{z^{2n-2}}{(2n-2)!} = \sum_{n=0}^{\infty} \dfrac{z^{2n}}{(2n)!} = f(z)$ by reindexing.
- Since $f''(z)$ also has radius of convergence ∞, this means $f''(z) = f(z)$ on all of $\mathbb{C}$.

(b) Show that $f(z) = \sum_{n=1}^{\infty} \dfrac{z^n}{n}$ satisfies $f'(z) = \dfrac{1}{1-z}$ for $|z| < 1$.

- First note that $f(z)$ has radius of convergence 1, as shown in problem 6(a) of homework 3.
- Then for $|z| < 1$ we have $f'(z) = \sum_{n=1}^{\infty} \dfrac{nz^{n-1}}{n} = \sum_{n=1}^{\infty} z^{n-1} = \sum_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$ since this last series is the usual geometric series which converges to $\dfrac{1}{1-z}$ for $|z| < 1$.

(c) Show that $f(z) = \sum_{n=0}^{\infty} \dfrac{z^{2n}}{(n!)^2}$ satisfies $z^2 f''(z) + zf'(z) = 4z^2 f(z)$ for all $z \in \mathbb{C}$.

- First, by the ratio test, with $a_{2n} = \dfrac{1}{(n!)^2}$, we see that $\dfrac{a_{2n+2}}{a_{2n}} = \dfrac{1}{(n+1)^2} \to 0$ as $n \to \infty$, so the series has radius of convergence ∞.
- Then differentiating term-by-term yields $f'(z) = \sum_{n=0}^{\infty} \dfrac{2nz^{2n-1}}{(n!)^2} = \sum_{n=1}^{\infty} \dfrac{2z^{2n-1}}{n!(n-1)!}$ so $f''(z) = \sum_{n=1}^{\infty} \dfrac{2(2n-1)z^{2n-2}}{n!(n-1)!}$.
- So then $z^2 f''(z) = \sum_{n=1}^{\infty} \dfrac{2(2n-1)z^{2n}}{n!(n-1)!}$ while $zf'(z) = \sum_{n=1}^{\infty} \dfrac{2z^{2n}}{n!(n-1)!}$, so the sum $z^2 f''(z) + zf'(z) = \sum_{n=1}^{\infty} \dfrac{2(2n)z^{2n}}{n!(n-1)!} = \sum_{n=1}^{\infty} \dfrac{4z^{2n}}{[(n-1)!]^2}$. But $4z^2 f(z) = \sum_{n=0}^{\infty} \dfrac{4z^{2n+2}}{(n!)^2} = \sum_{n=1}^{\infty} \dfrac{4z^{2n}}{[(n-1)!]^2}$, so we do have $z^2 f''(z) + zf'(z) = 4z^2 f(z)$. Since both sides have radius of convergence ∞, this means $f''(z) = f(z)$ on all of $\mathbb{C}$.

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5. We can often construct series solutions to a differential equation involving a function $f(z)$ by writing $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, differentiating formally, and then solving for the coefficients a_i. An advantage to this approach is that it can also provide an easy proof for uniqueness of the solution under the (often reasonable) assumption that the solution is analytic at $z = z_0$.

(a) Show that the unique analytic solution at $z = 0$ to $f'(z) = f(z)$ with $f(0) = 1$ is $f(z) = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = e^z$.

- Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic at $z = 0$ and has $f'(z) = f(z)$ and $f(0) = 1$.
- Then $f(0) = 1$ implies $a_0 = 1$, and then since $f'(z) = \sum_{n=0}^{\infty} na_n z^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} z^n$, we have $(n+1)a_{n+1} = a_n$ for each n.

- So, since $a_0 = 1$, we see successively that $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3!}$, and then by a trivial induction we have $a_n = \frac{1}{n!}$ for each n.
- Thus, the only possible solution is $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$. But since this function does satisfy all of the conditions, it is the unique analytic solution.

(b) Find the terms up to order 4 in a series expansion for an analytic solution at $z = 0$ to the differential equation $f''(z) + 3zf(z) = 1$ with $f(0) = f'(0) = 1$.

- Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic at $z = 0$ and has $f''(z) + 3zf(z) = 1$ and $f(0) = f'(0) = 1$.
- Then $f(0) = f'(0) = 1$ yields $a_0 = a_1 = 1$, so since $f''(z) = \sum_{n=0}^{\infty} n(n-1)a_n z^{n-1} = \sum_{n=0}^{\infty} n(n+1)a_{n+2}z^n$ and $3zf(z) = \sum_{n=0}^{\infty} 3a_n z^{n+1} = \sum_{n=0}^{\infty} 3a_{n-1}z^n$, comparing coefficients yields $\sum_{n=0}^{\infty}[3a_{n-1} + n(n+1)a_{n+2}]z^n = 1$.
- For $n = 0$ this yields $2a_2 = 1$ so $a_2 = \frac{1}{2}$, and for larger n we get $a_{n+2} = -\frac{3a_{n-1}}{n(n+1)}$.
- So starting with $a_0 = 1$, $a_1 = 1$, $a_2 = \frac{1}{2}$, we get $a_3 = -\frac{1}{2}$ and $a_4 = -\frac{1}{4}$, yielding $f(z) = \boxed{1 + z + \frac{1}{2}z^2 - \frac{1}{2}z^3 - \frac{1}{4}z^4 + \cdots}$.

(c) Find the terms up to order 4 in a series expansion for an analytic solution at $z = 1$ to the differential equation $f''(z) + f'(z) + (z-1)^2 f(z) = 0$ with $f(1) = f'(1) = 1$.

- Suppose $f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n$ is analytic at $z = 1$ and has $f''(z) + f'(z) + (z-1)^2 f(z) = 0$ and $f(1) = f'(1) = 1$. Set $w = z - 1$, so that the differential equation is $f''(w) + f'(w) + w^2 f(w) = 0$.
- Then $f(1) = f'(1) = 1$ yields $a_0 = a_1 = 1$, so since $f'(w) = \sum_{n=0}^{\infty} na_n w^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}w^n$, $f''(w) = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}w^n$, and $w^2 f(w) = \sum_{n=0}^{\infty} a_{n-2}w^n$, comparing coefficients yields $\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} + (n+1)a_{n+1} + a_{n-2}]w^n = 0$.
- For $n = 0$ we obtain $2a_2 + a_1 = 0$ so $a_2 = -\frac{1}{2}$. For $n = 1$ we obtain $6a_3 + 2a_2 = 0$ so $a_3 = \frac{1}{6}$. Finally, for $n \geq 2$ we have $a_{n+2} = -\frac{a_{n-2} + (n+1)a_{n+1}}{(n+1)(n+2)}$ which gives $a_4 = -\frac{1 + 3a_3}{3 \cdot 4} = -\frac{1}{8}$.
- Thus we obtain $f(z) = \boxed{1 + (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{6}(z-1)^3 - \frac{1}{8}(z-1)^4 + \cdots}$.

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6. The goal of this problem is to prove (versions of) L'Hôpital's Rule for $\frac{0}{0}$ limits of holomorphic and analytic functions.

(a) Suppose that f and g are holomorphic at $z = a$, and that $f(a) = g(a) = 0$ where $g'(a) \neq 0$. Prove that $\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)}$. [Hint: Evaluate $\frac{[f(z) - f(a)]/(z-a)}{[g(z) - g(a)]/(z-a)}$.]

- Per the hint, for $z \neq a$ we have $\frac{f(z)}{g(z)} = \frac{f(z) - f(a)}{g(z) - g(a)} = \frac{[f(z) - f(a)]/(z-a)}{[g(z) - g(a)]/(z-a)}$.
- Taking the limit as $z \to a$ then yields $\lim_{z \to a} \frac{f(z)}{g(z)} = \lim_{z \to a} \frac{[f(z) - f(a)]/(z-a)}{[g(z) - g(a)]/(z-a)}$, but since the limit of the numerator is $f'(a)$ and the limit of the denominator is $g'(a) \neq 0$, the limit of the quotient is the quotient of the limits, namely $f'(a)/g'(a)$.

(b) Suppose that f and g are analytic at $z = a$ and that $f(a) = g(a) = 0$ where g has order d at $z = a$ (i.e., the first nonzero coefficient in the series expansion for g is the coefficient of $(z-a)^d$). Prove that $\lim_{z \to a} \frac{f(z)}{g(z)}$ exists if and only if f has order at least d at $z = a$, and in such a case, $\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f^{(d)}(a)}{g^{(d)}(a)}$. [Hint: Restrict attention to a disc of positive radius where f and g have their only zero at $z = a$.]

- If f is identically zero the result is trivial. Otherwise, as noted in class, by the uniqueness result there exists a disc of positive radius centered at a such that f has its only zero at $z = a$ inside the disc, and likewise there exists such a disc for g. Let R be the smaller of the radii of these two discs.
- Now let $f(z) = a_e(z-a)^e + a_{e+1}(z-a)^{e+1} + \cdots$ and $g(z) = b_d(z-a)^d + b_{d+1}(z-a)^{d+1} + \cdots$, where both expansions converge on the disc $|z-a| < R$ for some positive R, and $f(z), g(z) \neq 0$ except at $z = a$.
- Then for $z \neq a$ in the disc, we have $\dfrac{f(z)}{g(z)} = \dfrac{a_e(z-a)^e + a_{e+1}(z-a)^{e+1} + \cdots}{b_d(z-a)^d + b_{d+1}(z-a)^{d+1} + \cdots} = (z-a)^{e-d}\dfrac{a_e + a_{e+1}(z-a) + \cdots}{b_d + b_{d+1}(z-a) + \cdots}$.
- As $z \to a$, the second quantity has limit $\dfrac{a_e}{b_d}$ because the two corresponding power series are both continuous at $z = a$.
- Therefore, the limit exists if and only if $\lim_{z\to a}(z-a)^{e-d}$ exists, and that is clearly true only when $e \geq d$: namely, when f vanishes to order at least d.
- In that case, the value of the limit equals 0 if $e > d$ (since the exponent on $z - a$ is positive) and in that case $f^{(d)}(a) = 0$ so the limit value is $\dfrac{f^{(d)}(a)}{g^{(d)}(a)}$, and otherwise if $e = d$ then the value of the limit is $\dfrac{a_e}{b_d} = \dfrac{a_d}{b_d} = \dfrac{f^{(d)}(a)/d!}{g^{(d)}(a)/d!} = \dfrac{f^{(d)}(a)}{g^{(d)}(a)}$ as claimed.

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7. [Challenge] The goal of this problem is to give an integral whose evaluation via Riemann sums is actually easier than most other approaches. Fix $a > 1$ and consider the integral $I_a = \int_0^\pi \ln(a^2 - 2a\cos x + 1)\, dx$.

(a) Show that $\prod_{k=1}^{n-1}(a^2 - 2a\cos\dfrac{k\pi}{n} + 1) = \dfrac{a^{2n}-1}{a^2-1}$. [Hint: Factor in $\mathbb{C}$.]

- First observe that $a^2 - 2a\cos\dfrac{k\pi}{n} + 1 = (a - e^{2\pi ki/n})(a - e^{-2\pi ki/n})$ by direct expansion.
- Therefore, $\prod_{k=1}^{n-1}(a^2 - 2a\cos\dfrac{k\pi}{n} + 1) = \prod_{k=1}^{n-1}(a - e^{2\pi ki/n})(a - e^{-2\pi ki/n})$.
- If we now multiply this expression by $a^2 - 1 = (a - e^{0\pi i/n})(a - e^{2n\pi i/n})$, we obtain the product $\prod_{j=0}^{2n-1}(a - e^{2\pi ji/n})$, which is the factorization of $a^{2n} - 1$.
- Thus we have $\prod_{k=1}^{n-1}(a^2 - 2a\cos\dfrac{k\pi}{n} + 1) = \dfrac{a^{2n}-1}{a^2-1}$ as claimed.

(b) Show that the right-endpoint Riemann sum for I_a with partition $P = \{0, \pi/n, 2\pi/n, \ldots, \pi(n-1)/n, \pi\}$ is equal to $\dfrac{\pi}{n}\ln\dfrac{a^{2n}-1}{a^2-1} + \dfrac{\pi}{n}\ln(a+1)^2$.

- Since each interval has width π/n the Riemann sum is $\dfrac{\pi}{n}\sum_{k=1}^{n}\ln(a^2 - 2a\cos\dfrac{k\pi}{n} + 1)$.
- By (a), we have $\prod_{k=1}^{n-1}(a^2 - 2a\cos\dfrac{k\pi}{n} + 1) = \dfrac{a^{2n}-1}{a^2-1}$, so since logarithms turn products into sums, we see that $\sum_{k=1}^{n-1}\ln(a^2 - 2a\cos\dfrac{k\pi}{n} + 1) = \ln[\prod_{k=1}^{n-1}(a^2 - 2a\cos\dfrac{k\pi}{n} + 1)] = \ln\dfrac{a^{2n}-1}{a^2-1}$.
- Thus after taking out the last term, the Riemann sum simplifies to $\dfrac{\pi}{n}\ln\dfrac{a^{2n}-1}{a^2-1} + \dfrac{\pi}{n}\ln(a+1)^2$.

(c) Show that the value of the integral I_a is $2\pi\ln a$.

- Since $a^2 - 2a\cos x + 1 \geq a^2 - 2a + 1 = (a-1)^2 > 0$ the integral is well defined hence equals the limit of its Riemann sums.
- The Riemann sum equals $\dfrac{\pi}{n}\ln\dfrac{a^{2n}-1}{a^2-1} + \dfrac{\pi}{n}\ln(a+1)^2 = \dfrac{\pi}{n}\ln(a^{2n}-1) + \dfrac{\pi}{n}\ln\dfrac{a+1}{a-1}$ from (b).
- The limit of the second term is zero, while the limit of the first term we can compute using L'Hôpital's rule: $\lim_{n\to\infty}\dfrac{\pi\ln(a^{2n}-1)}{n} = \lim_{n\to\infty}\dfrac{\pi a^{2n}\ln(a^2)}{a^{2n}-1} = \lim_{n\to\infty}\dfrac{2\pi\ln a}{1-a^{-2n}} = 2\pi\ln a$. So the limit equals $2\pi\ln a$ as claimed.
