

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find all possible complex values for each of these expressions (note that $\log$ always denotes the multivalued complex logarithm):

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|-----------------|---------------------|----------------------|
| (a) $\log(e)$. | (e) $e^{\log(i)}$. | (i) 1^2 . |
| (b) $\log(i)$. | (f) $\log(e^i)$. | (j) $(1^2)^{1/6}$. |
| (c) i^{2i} . | (g) $1^{1/6}$. | (k) $1^{2/6}$. |
| (d) 4^i . | (h) $(1^{1/6})^2$. | (l) $(-1)^{1/\pi}$. |
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2. For each function f on each curve γ , calculate $\int_{\gamma} f(z) dz$:

- (a) $f(z) = 1/z$, $\gamma(t) = e^{it}$ for $0 \leq t \leq \pi/2$.
 - (b) $f(z) = z^n$ (n an integer with $n \neq -1$), $\gamma(t) = e^{it}$ for $0 \leq t \leq \pi/2$.
 - (c) $f(z) = 3z^2$, $\gamma(t) = t + (1-t)i$ for $0 \leq t \leq 1$.
 - (d) $f(z) = 3\bar{z}^2$, $\gamma(t) = t + (1-t)i$ for $0 \leq t \leq 1$.
 - (e) $f(z) = \text{Log}(z)$, $\gamma(t) = 2e^{it}$ for $0 \leq t \leq 2\pi$, where (as usual) Log denotes the principal complex logarithm.
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3. For each contour γ , give a parametrization and then evaluate $\int_{\gamma} f(z) dz$:

- (a) The circle $|z - 1| = 2$ traversed twice counterclockwise, where $f(z) = \bar{z}$.
 - (b) The path that starts at 0, follows a straight line to $1 + 2i$, and then follows a straight line to $4 + 2i$, where $f(z) = z$.
 - (c) The counterclockwise boundary of the triangle with vertices 0, 2, and $1 + i$, where $f(z) = \text{Re}(z)$.
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Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. By differentiating a power series expansion, we can often show it satisfies a differential equation inside its radius of convergence. Show the following (make sure to check the radius of convergence!):

- (a) Show that $f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ satisfies $f''(z) = f(z)$ for all $z \in \mathbb{C}$.
 - (b) Show that $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ satisfies $f'(z) = \frac{1}{1-z}$ for $|z| < 1$.
 - (c) Show that $f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(n!)^2}$ satisfies $z^2 f''(z) + z f'(z) = 4z^2 f(z)$ for all $z \in \mathbb{C}$.
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5. We can often construct series solutions to a differential equation involving a function $f(z)$ by writing $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, differentiating formally, and then solving for the coefficients a_i . An advantage to this approach is that it can also provide an easy proof for uniqueness of the solution under the (often reasonable) assumption that the solution is analytic at $z = z_0$.

- (a) Show that the unique analytic solution at $z = 0$ to $f'(z) = f(z)$ with $f(0) = 1$ is $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.
 - (b) Find the terms up to order 4 in a series expansion for an analytic solution at $z = 0$ to the differential equation $f''(z) + 3zf(z) = 1$ with $f(0) = f'(0) = 1$.
 - (c) Find the terms up to order 4 in a series expansion for an analytic solution at $z = 1$ to the differential equation $f''(z) + f'(z) + (z - 1)^2 f(z) = 0$ with $f(1) = f'(1) = 1$.
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6. The goal of this problem is to prove (versions of) L'Hôpital's Rule for $\frac{0}{0}$ limits of holomorphic and analytic functions.

- (a) Suppose that f and g are holomorphic at $z = a$, and that $f(a) = g(a) = 0$ where $g'(a) \neq 0$. Prove that $\lim_{z \rightarrow a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)}$. [Hint: Evaluate $\frac{[f(z) - f(a)]/(z - a)}{[g(z) - g(a)]/(z - a)}$.]
 - (b) Suppose that f and g are analytic at $z = a$ and that $f(a) = g(a) = 0$ where g has order d at $z = a$ (i.e., the first nonzero coefficient in the series expansion for g is the coefficient of $(z - a)^d$). Prove that $\lim_{z \rightarrow a} \frac{f(z)}{g(z)}$ exists if and only if f has order at least d at $z = a$, and in such a case, $\lim_{z \rightarrow a} \frac{f(z)}{g(z)} = \frac{f^{(d)}(a)}{g^{(d)}(a)}$. [Hint: Restrict attention to a disc of positive radius where f and g have their only zero at $z = a$.]
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7. [Challenge] The goal of this problem is to give an integral whose evaluation via Riemann sums is actually easier than most other approaches. Fix $a > 1$ and consider the integral $I_a = \int_0^\pi \ln(a^2 - 2a \cos x + 1) dx$.

- (a) Show that $\prod_{k=1}^{n-1} (a^2 - 2a \cos \frac{k\pi}{n} + 1) = \frac{a^{2n} - 1}{a^2 - 1}$. [Hint: Factor in $\mathbb{C}$.]
 - (b) Show that the right-endpoint Riemann sum for I_a with partition $P = \{0, \pi/n, 2\pi/n, \dots, \pi(n-1)/n, \pi\}$ is equal to $\frac{\pi}{n} \ln \frac{a^{2n} - 1}{a^2 - 1} + \frac{\pi}{n} \ln(a + 1)^2$.
 - (c) Show that the value of the integral I_a is $2\pi \ln a$.
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