1. Find the radius and the disc of convergence for each power series:

   (a) $\sum_{n=0}^{\infty}(z-1+i)^n$.

   - Setting $w = z-1+i = z-(1-i)$ we get the geometric series $\sum_{n=0}^{\infty} w^n$ which has radius of convergence $\boxed{1}$.
   - The center is $z_0 = 1-i$ so the disc is $|z-(1-i)| < 1$.

   (b) $\sum_{n=0}^{\infty} \dfrac{(z-i)^n}{n!}$.

   - Setting $w = z-i$ we get the exponential series $\sum_{n=0}^{\infty} \dfrac{w^n}{n!}$ which has radius of convergence $\boxed{\infty}$. So the disc of convergence is simply $\boxed{\mathbb{C}}$.

   (c) $\sum_{n=1}^{\infty} n^n (z-1)^n$.

   - Setting $w = z-1$ we get the series $\sum_{n=1}^{\infty} n^n w^n$ which from 1(c) of homework 3 has radius of convergence $\boxed{0}$. So the disc of convergence is simply the point $\boxed{z=1}$.

   (d) $\sum_{n=1}^{\infty} \dfrac{(z+2)^n}{n^n}$.

   - Setting $w = z+2$ we get the series $\sum_{n=1}^{\infty} \dfrac{w^n}{n^n}$ which from 1(b) of homework 3 has radius of convergence ∞. So the disc of convergence is simply $\boxed{\mathbb{C}}$.

   (e) $\sum_{n=0}^{\infty}(2z+1)^n$.

   - Setting $w = 2z+1$ we get the geometric series $\sum_{n=0}^{\infty} w^n$ which converges for $|w| < 1$. This yields $|2z+1| < 1$ hence the disc of convergence is $\boxed{\left|z+\dfrac{1}{2}\right| < \dfrac{1}{2}}$ with radius $\boxed{\dfrac{1}{2}}$.

   (f) $\sum_{n=1}^{\infty} \dfrac{\pi^n}{n^e}(\pi z+e)^n$.

   - Rearranging yields the series $\sum_{n=1}^{\infty} \dfrac{\pi^{2n}}{n^e}(z+e/\pi)^n$. Setting $w = z+e/\pi$ yields the series $\sum_{n=1}^{\infty} \dfrac{\pi^{2n}}{n^e} w^n$.
   - Since $\lim_{n\to\infty} \left|\dfrac{\pi^{2n}}{n^e}\right|^{1/n} = \lim_{n\to\infty} \dfrac{\pi^2}{n^{e/n}} = \pi^2$ we see that the series has radius of convergence $\boxed{1/\pi^2}$, so the disc of convergence is $\boxed{\left|z+\dfrac{e}{\pi}\right| < \dfrac{1}{\pi^2}}$.

   (g) $\sum_{n=0}^{\infty} \cosh(n)\cdot z^n$.

   - Note that $\cosh(n) = \dfrac{e^n+e^{-n}}{2}$. Then $\lim_{n\to\infty} |\cosh(n)|^{1/n} = \lim_{n\to\infty} [e^n + e^{-n}]^{1/n} 2^{-1/n} = e$ since the term $e^{-n} \to 0$.
   - So the radius of convergence is $\boxed{1/e}$ and the disc of convergence is $\boxed{|z| < 1/e}$.

   (h) $\sum_{n=0}^{\infty} \dfrac{(3z+i)^{3n}}{(2-i)^n}$.

   - Rearranging yields the geometric series $\sum_{n=0}^{\infty} \left[\dfrac{(3z+i)^3}{2-i}\right]^n$ which converges for $\left|\dfrac{(3z+i)^3}{2-i}\right| < 1$.
   - Equivalently, this is $|3z+i| < 5^{1/6}$ which is the disc $\boxed{|z+i/3| < \dfrac{5^{1/6}}{3}}$ of radius $\boxed{\dfrac{5^{1/6}}{3}}$.

2. Find power series expansions for each given function $f(z)$ centered at the given point $z = z_0$:

   (a) $f(z) = \dfrac{z}{1-z^3}$ around $z = 0$. [Hint: Use $\dfrac{1}{1-r} = \sum_{n=0}^{\infty} r^n$.]

   - Using the geometric series expansion $\dfrac{1}{1-r} = \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$ with $r = z^3$ yields $\dfrac{1}{1-z^3} = \sum_{n=0}^{\infty} z^{3n}$. So we get $f = \boxed{\sum_{n=0}^{\infty} z^{3n+1} = z + z^4 + z^7 + z^{10} + \cdots}$.

   (b) $f(z) = 1 + z + z^2 + z^4$ around $z = -2$.

   - Setting $w = z + 2$ so that $z = w - 2$ yields $f(z) = 1 + (w-2) + (w-2)^2 + (w-2)^4 = 19 - 35w + 25w^2 - 8w^3 + w^4$.
   - So we get $f = \boxed{19 - 35(z+2) + 25(z+2)^2 - 8(z+2)^3 + (z+2)^4}$.
   - Alternatively, $f(-2) = 19$, $f'(-2) = -35$, $f''(-2) = 50$, $f'''(-2) = -48$, $f''''(-2) = 24$, and $f^{(n)}(-2) = 0$ for $n \geq 5$. So using $a_n = f^{(n)}(-2)/n!$ yields $f = \boxed{19 - 35(z+2) + 25(z+2)^2 - 8(z+2)^3 + (z+2)^4}$

   (c) $f(z) = (1+z)/(1-z)$ around $z = -1$.

   - Setting $w = z + 1$ so that $z = w - 1$ yields $f(z) = \dfrac{w}{2-w} = \dfrac{w}{2} \cdot \dfrac{1}{1-(w/2)} = \dfrac{w}{2} \cdot \sum_{n=0}^{\infty} (w/2)^n = \sum_{n=1}^{\infty} \dfrac{w^n}{2^n}$. So we get $f = \boxed{\sum_{n=1}^{\infty} \dfrac{(z+1)^n}{2^n} = \dfrac{z+1}{2} + \dfrac{(z+1)^2}{2^2} + \dfrac{(z+1)^3}{2^3} + \cdots}$.

   (d) $f(z) = e^z$ around $z = i$.

   - Using the formula $a_n = f^{(n)}(i)/n!$, and the fact that $f^{(n)}(z) = e^z$, we see $a_n = e^i/n!$.
   - So we get $f = \boxed{\sum_{n=0}^{\infty} \dfrac{e^i}{n!}(z-i)^n = 1 + e^i(z-i) + \dfrac{e^i}{2!}(z-i)^2 + \dfrac{e^i}{3!}(z-i)^3 + \cdots}$.
   - Equivalently, noting $e^i = \cos 1 + i \sin 1$, this is also $\sum_{n=0}^{\infty} \dfrac{\cos 1 + i \sin 1}{n!}(z-i)^n$.

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3. Find all solutions $z \in \mathbb{C}$ to each of the following equations:

   (a) $e^{4z} = i$.

   - Since $i = e^{i\pi/2}$ we have $e^{4z} = e^{i\pi/2}$. As noted in class we have $e^z = e^w$ if and only if $z - w = 2k\pi i$ for some integer k.
   - So this yields $4z - i\pi/2 = 2k\pi i$ for some integer k, whence $z = \boxed{(\pi/8 + k\pi/2)i}$ for some integer k.

   (b) $e^{iz} = 4$.

   - Like in (a) we have $e^{iz} = e^{\ln(4)}$, so $iz - \ln(4) = 2k\pi i$ for some integer k.
   - This yields $z = \boxed{-i\ln(4) + 2k\pi}$ for some integer k.

   (c) $\cosh(z) = 5/4$.

   - By definition we have $\cosh(z) = \dfrac{e^z + e^{-z}}{2}$ so we must have $e^z + e^{-z} = 5/2$.
   - Setting $w = e^z$ yields $w + w^{-1} = 5/2$ so that $w^2 - \dfrac{5}{2}w + 1 = 0$ yielding $w = 2, 1/2$. This yields $z = \boxed{\pm\ln(2) + 2k\pi i}$ for some integer k.

   (d) $\cos(z) = 5/4$.

   - Note that $\cos(z) = \cosh(iz)$ so from (c) we have $iz = \pm\ln(2) + 2k\pi i$ for some integer k. Thus we get $z = \boxed{2k\pi \pm i\ln(2)}$ for some integer k.

(e) $\sinh(z) = i\cosh(z)$.

- The equation yields $\dfrac{e^z - e^{-z}}{2} = i\dfrac{e^z + e^{-z}}{2}$. Setting $w = e^z$ and multiplying by 2 yields $w^2 - 1 = i(w^2 + 1)$ so $w^2 = i$.
- Thus we get $e^{2z} = i = e^{\pi i/2}$ so as in (a) we obtain $2z = \pi i/2 + 2k\pi i$ hence $z = \boxed{(\pi/4 + k\pi)i}$ for some integer k.

(f) $\sinh(z) = \cosh(z)$.

- The equation yields $\dfrac{e^z - e^{-z}}{2} = \dfrac{e^z + e^{-z}}{2}$ which reduces to $e^{-z} = 0$. This has $\boxed{\text{no solutions}}$.

(g) $\sin(z) = i\cos(z)$.

- Since $\cos(z) = \cosh(iz)$ and $\sin(z) = i\sinh(iz)$ the equation yields $i\sinh(iz) = i\cosh(iz)$ hence $\sinh(iz) = \cosh(iz)$ which by (f) has $\boxed{\text{no solutions}}$.

(h) $\sinh(z) = (1 + 3i)/4$.

- By definition we have $\sinh(z) = \dfrac{e^z - e^{-z}}{2}$ so we must have $e^z - e^{-z} = \dfrac{1+3i}{2}$.
- Setting $w = e^z$ yields $w - w^{-1} = \dfrac{1+3i}{2}$ so that $w^2 - \dfrac{1+3i}{2}w - 1 = 0$ yielding $w = 1 + i, \dfrac{-1+i}{2}$, which in polar form are $\sqrt{2}e^{i\pi/4}$ and $\dfrac{1}{\sqrt{2}}e^{3i\pi/4}$.
- This yields $z = \boxed{\ln(\sqrt{2}) + (\pi/4 + 2k\pi)i, \ -\ln(\sqrt{2}) + (3\pi/4 + 2k\pi)i}$ for some integer k.

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4. Prove the following things about the complex exponential and (hyperbolic) trigonometric functions:

(a) Show $\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$ and $\cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$.

- Note that $\sinh(t) = i\sin(it)$ and $\cosh(t) = \cos(it)$.
- From the sine addition formula we have $\sin(x+iy) = \sin(x)\cos(iy)+\cos(x)\sin(iy) = \sin(x)\cosh(y)+ i\cos(x)\sinh(y)$.
- From the cosine addition formula we have $\cos(x+iy) = \cos(x)\cos(iy)-\sin(x)\sin(iy) = \cos(x)\cosh(y)- i\sin(x)\sinh(y)$.

(b) Show $\sinh(z+w) = \sinh(z)\cosh(w)+\cosh(z)\sinh(w)$ and $\cosh(z+w) = \cosh(z)\cosh(w)+\sinh(z)\sinh(w)$.

- Using the sine addition formula $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$ yields $\sinh(z + w) = i\sinh(iz + iw) = i\sin(iz)\cos(iw) + i\cos(iz)\sin(iw) = \sinh(z)\cosh(w) + \cosh(z)\sinh(w)$.
- Using the cosine addition formula $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$ yields $\cosh(z + w) = \cos(iz + iw) = \cos(iz)\cos(iw) - \sin(iz)\sin(iw) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$.

(c) Show $\tanh(z + w) = \dfrac{\tanh(z) + \tanh(w)}{1 + \tanh(z)\tanh(w)}$. Deduce that $\tanh(z)$ is periodic with period $i\pi$.

- By (b) we have $\tanh(z+w) = \dfrac{\sinh(z + w)}{\cosh(z + w)} = \dfrac{\sinh(z)\cosh(w) + \cosh(z)\sinh(w)}{\cosh(z)\cosh(w) + \sinh(z)\sinh(w)} = \dfrac{\tanh(z) + \tanh(w)}{1 + \tanh(z)\tanh(w)}$ after dividing the top and bottom by $\cosh(z)\cosh(w)$.
- Setting $w = i\pi$ and noting $\tanh(i\pi) = \dfrac{\sinh(i\pi)}{\cosh(i\pi)} = 0$ yields $\tanh(z + i\pi) = \dfrac{\tanh(z) + 0}{1 + 0} = \tanh(z)$, so $\tanh(z)$ is periodic with period $i\pi$.

(d) Show e^z is one-to-one (in other words, that $e^z = e^w$ implies $z = w$) on any open disc of radius π.

- As noted in class we have $e^z = e^w$ if and only if $z - w = 2k\pi i$ for some integer k.
- However, if z, w both lie in an open disc of radius π, then $|z - w| < 2\pi$. Thus if $e^z = e^w$ this forces $2\pi\,|k| < 2\pi$ hence $k = 0$ hence $z = w$. Thus e^z is one-to-one as claimed.

(e) Show $2\cos(\dfrac{z+w}{2})\sin(\dfrac{z-w}{2}) = \sin(z) - \sin(w)$. Deduce that $\sin(z) = \sin(w)$ if and only if $z + w = (2k + 1)\pi$ or $z - w = 2k\pi$ for an integer k.

- For $\alpha = \frac{z+w}{2}$ and $\beta = \frac{z-w}{2}$ then $\sin(z) - \sin(w) = \sin(\alpha+\beta) - \sin(\alpha-\beta) = [\sin\alpha\cos\beta + \cos\alpha\sin\beta] - [\sin\alpha\cos\beta - \cos\alpha\sin\beta] = 2\cos\alpha\sin\beta = 2\cos(\frac{z+w}{2})\sin(\frac{z-w}{2})$ as claimed.
- For the second part by the identity we see $\sin(z) = \sin(w)$ if and only if $\cos(\frac{z+w}{2}) = 0$ or $\sin(\frac{z-w}{2})$.
- By the characterization of the complex zeroes of sine and cosine (i.e., just the real zeroes) these are equivalent to $\frac{z+w}{2} = \frac{\pi}{2} + k\pi$ and $\frac{z-w}{2} = k\pi$ for some integer k, which are in turn equivalent to $z + w = (2k+1)\pi$ and $z - w = 2k\pi$.
- So we see $\sin(z) = \sin(w)$ if and only if $z + w = (2k+1)\pi$ or $z - w = 2k\pi$ for an integer k as required.

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5. Let F_n be the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. (The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, ...) The goal of this problem is to study the power series $f(z) = \sum_{n=0}^{\infty} F_n z^n$, the <u>generating function</u> for the Fibonacci numbers.

  (a) Show that $(1 - z - z^2)f(z) = z$ as a formal power series and deduce $f(z) = \dfrac{z}{1 - z - z^2}$.

  - We have $(1 - z - z^2)f(z) = \sum_{n=0}^{\infty} F_n z^n - \sum_{n=0}^{\infty} F_n z^{n+1} - \sum_{n=0}^{\infty} F_n z^{n+2} = F_0 + (F_1 - F_0)z + \sum_{n=2}^{\infty}(F_n - F_{n-1} - F_{n-2})z^n = 0 + z + \sum_{n=2}^{\infty} 0z^n = z$.
  - Thus $f(z) = \dfrac{z}{1 - z - z^2}$ as a formal power series.

  (b) Find complex constants a, α, b, β such that $\dfrac{z}{1 - z - z^2} = \dfrac{a}{1 - \alpha z} + \dfrac{b}{1 - \beta z}$.

  - Summing the right-hand side gives $\dfrac{a(1 - \beta z) + b(1 - \alpha z)}{(1 - \alpha z)(1 - \beta z)}$, so we want $(1 - \alpha z)(1 - \beta z) = 1 - z - z^2$ and $a(1 - \beta z) + b(1 - \alpha z) = z$.
  - Since $1 - z - z^2$ has roots $z = \dfrac{-1 \pm \sqrt{5}}{2}$ taking reciprocals yields the factorization $1 - z - z^2 = (1 - \dfrac{1+\sqrt{5}}{2}z)(1 - \dfrac{1-\sqrt{5}}{2}z)$, so we want $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.
  - We also want $a(1 - \beta z) + b(1 - \alpha z) = z$ which is equivalent to $a + b = 0$ and $\beta a + \alpha b = -1$. So $b = -a$ and then $a = \dfrac{1}{\beta - \alpha} = \dfrac{1}{\sqrt{5}}$ with $b = -\dfrac{1}{\sqrt{5}}$.

  (c) Prove Binet's formula for the Fibonacci numbers: $F_n = \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$ where $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \dfrac{1-\sqrt{5}}{2}$. [Hint: Expand the two geometric series from (b) and compare to $f(z)$.]

  - From (b), $\dfrac{z}{1 - z - z^2} = \dfrac{1/\sqrt{5}}{1 - \varphi z} + \dfrac{-1/\sqrt{5}}{1 - \overline{\varphi} z} = \dfrac{1}{\sqrt{5}} \cdot \sum_{n=0}^{\infty} \varphi^n z^n - \dfrac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \overline{\varphi}^n z^n = \sum_{n=0}^{\infty} \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}} z^n$.
  - But from (a) we know that $\dfrac{z}{1 - z - z^2} = \sum_{n=0}^{\infty} F_n z^n$. So comparing coefficients immediately yields the desired formula.

  (d) Find the radius of convergence of $f(z)$.

  - We have $\lim_{n\to\infty} |F_n|^{1/n} = \lim_{n\to\infty} \left| \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}} \right|^{1/n} = \varphi$ since $\varphi > 1$ while $-1 < \overline{\varphi} < 0$.
  - So by the radius-of-convergence formula, the radius is $R = 1/\varphi = \boxed{\dfrac{-1+\sqrt{5}}{2}}$.
  - Alternatively, we could use the result of problem 4 of homework 3: the series $\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \varphi^n z^n$ has radius of convergence $1/\varphi$ while the series $-\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \overline{\varphi}^n z^n$ has radius of convergence $1/|\overline{\varphi}|$, so their sum has radius of convergence the minimum of these, which is $1/\varphi$.

Remark: A similar method to the one in (a)-(c) can be used to solve any linear recurrence with constant coefficients, of the form $a_{n+1} = c_n a_n + \cdots + c_{n-k} a_{n-k}$ for constants c_i. Moreover, the general technique of considering the generating function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for an arbitrary sequence $a_0, a_1, \ldots$ can be used to find and prove many kinds of combinatorial identities.

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6. The goal of this problem is to prove that if p is any polynomial, then the formal power series $\sum_{n=0}^{\infty} p(n)z^n$ is a rational function in z.

   (a) Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that $zf'(z) = \sum_{n=0}^{\infty} na_n z^n$.

   - This is immediate from the termwise differentiation formula, since $f'(z) = \sum_{n=0}^{\infty} na_n z^{n-1}$, so then multiplying by z gives $zf'(z) = \sum_{n=0}^{\infty} na_n z^n$.

   (b) Show that for every integer $k \geq 0$, $\sum_{n=0}^{\infty} n^k z^n$ is a rational function in z. [Hint: Induct on k.]

   - We induct on k. For the base case $k = 0$ we have $\sum_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$ as shown in class.
   - For the inductive step, suppose $f(z) = \sum_{n=0}^{\infty} n^k z^n$ is a rational function in z. Then by (a) we have $\sum_{n=0}^{\infty} n^{k+1} z^n = zf'(z)$, which is also a rational function in z, as required.

   (c) Show that $\sum_{n=0}^{\infty} p(n)z^n$ is a rational function in z for any polynomial $p(x) = b_d x^d + \cdots + b_0$.

   - If $p(x) = b_d x^d + \cdots + b_0$ then $\sum_{n=0}^{\infty}(b_d n^d + \cdots + b_0)z^n = \sum_{l=0}^{d} b_l \left[\sum_{n=0}^{\infty} n^d z^n\right]$, which is a finite sum of rational functions by part (b).

   (d) Express $\sum_{n=0}^{\infty}(2n+5)z^n$ and $\sum_{n=0}^{\infty}(n^2+n)z^n$ as rational functions in z.

   - Using the technique suggested by (a)-(c) we first compute $\sum_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$, so differentiating and then multiplying by z yields $\sum_{n=0}^{\infty} nz^n = \dfrac{z}{(1-z)^2}$ and then $\sum_{n=0}^{\infty} n^2 z^n = \dfrac{z(z+1)}{(1-z)^3}$.
   - Thus $\sum_{n=0}^{\infty}(2n+5)z^n = \boxed{\dfrac{5-3z}{(1-z)^2}}$ and $\sum_{n=0}^{\infty}(n^2+n)z^n = \boxed{\dfrac{2z}{(1-z)^3}}$.

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7. [Challenge] The goal of this problem is to study the complex analogue of Newton's binomial series. Let α be any complex number that is not a nonnegative integer. Define the <u>binomial coefficient</u> $\dbinom{\alpha}{n} = \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$ for each integer $n \geq 0$. Now define the <u>binomial series</u> $B_\alpha(z) = \sum_{n=0}^{\infty} \dbinom{\alpha}{n} z^n$. In 1665, Newton proved that if α is real, then $B_\alpha(x) = (1+x)^\alpha$ for all real $-1 < x < 1$.

   (a) Show that the radius of convergence of the binomial series equals 1. [Hint: Use the Ratio Test.]

   - We have $\dbinom{\alpha}{n+1} \Big/ \dbinom{\alpha}{n} = \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{(n+1)!} \Big/ \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} = \dfrac{\alpha-n}{n+1}$.
   - So $\lim_{n\to\infty} \left|\dbinom{\alpha}{n+1} \Big/ \dbinom{\alpha}{n}\right| = \lim_{n\to\infty}\left|\dfrac{\alpha-n}{n+1}\right| = \left|\lim_{n\to\infty}\dfrac{\alpha-n}{n+1}\right| = |-1| = 1$.
   - Thus by the ratio test, the series has radius of convergence 1.
   - <u>Remark</u>: Note that we are implicitly using the fact that α is not a nonnegative integer here because otherwise the binomial coefficients are eventually zero, in which case the ratio calculation is invalid.

   (b) For a positive integer m, show that $(B_{1/m}(z))^m = 1 + z$ for all $|z| < 1$. [Hint: Use Newton's binomial theorem and the uniqueness of series expansions.]

   - If x is real, then $B_{1/m}(x) = (1+z)^{1/m}$ for $-1 < x < 1$. So taking the mth power yields $(B_{1/m}(x))^m = 1 + x$ for such x.
   - This means that the two power series $(B_{1/m}(z))^m$ and $1+z$ agree for all real x with $-1 < x < 1$. But by our uniqueness result, since the difference is a power series with positive radius of convergence (namely, radius 1 by part (a)) with a sequence of zeroes that has limit 0, the difference is identically zero.
   - Thus in fact $(B_{1/m}(z))^m = 1 + z$ for all $|z| < 1$, as claimed.

   (c) Deduce that for $|z| < 1$, the binomial series $B_{1/m}(z)$ is a holomorphic function of z whose value is an mth root of $1 + z$.

   - By (a) the series $B_{1/m}(z)$ has radius of convergence 1, so it defines a holomorphic function for $|z| < 1$.
   - Furthermore, by (b) the mth power of this series is $1 + z$, so the value is an mth root of $1 + z$.
