E. Dummit's Math 4555 ~ Complex Analysis, Fall 2025 ~ Homework 4, due Fri Oct 3rd.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the radius and the disc of convergence for each power series:

   (a) $\sum_{n=0}^{\infty} (z-1+i)^n$.

   (b) $\sum_{n=0}^{\infty} \frac{(z-i)^n}{n!}$.

   (c) $\sum_{n=1}^{\infty} n^n (z-1)^n$.

   (d) $\sum_{n=1}^{\infty} \frac{(z+2)^n}{n^n}$.

   (e) $\sum_{n=0}^{\infty} (2z+1)^n$.

   (f) $\sum_{n=1}^{\infty} \frac{\pi^n}{n^e} (\pi z + e)^n$.

   (g) $\sum_{n=0}^{\infty} \cosh(n) \cdot z^n$.

   (h) $\sum_{n=0}^{\infty} \frac{(3z+i)^{3n}}{(2-i)^n}$.

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2. Find power series expansions for each given function $f(z)$ centered at the given point $z = z_0$:

   (a) $f(z) = \frac{z}{1-z^3}$ around $z = 0$. [Hint: Use $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$.]

   (b) $f(z) = 1 + z + z^2 + z^4$ around $z = -2$.

   (c) $f(z) = (1+z)/(1-z)$ around $z = -1$.

   (d) $f(z) = e^z$ around $z = i$.

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3. Find all solutions $z \in \mathbb{C}$ to each of the following equations:

   (a) $e^{4z} = i$.

   (b) $e^{iz} = 4$.

   (c) $\cosh(z) = 5/4$.

   (d) $\cos(z) = 5/4$.

   (e) $\sinh(z) = i\cosh(z)$.

   (f) $\sinh(z) = \cosh(z)$.

   (g) $\sin(z) = i\cos(z)$.

   (h) $\sinh(z) = (1+3i)/4$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Prove the following things about the complex exponential and (hyperbolic) trigonometric functions:

   (a) Show $\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$ and $\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$.

   (b) Show $\sinh(z+w) = \sinh(z)\cosh(w) + \cosh(z)\sinh(w)$ and $\cosh(z+w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$.

   (c) Show $\tanh(z+w) = \frac{\tanh(z)+\tanh(w)}{1+\tanh(z)\tanh(w)}$. Deduce that $\tanh(z)$ is periodic with period $i\pi$.

   (d) Show e^z is one-to-one (in other words, that $e^z = e^w$ implies $z = w$) on any open disc of radius π.

   (e) Show $2\cos(\frac{z+w}{2})\sin(\frac{z-w}{2}) = \sin(z) - \sin(w)$. Deduce that $\sin(z) = \sin(w)$ if and only if $z + w = (2k+1)\pi$ or $z - w = 2k\pi$ for an integer k.

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5. Let F_n be the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. (The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, ...) The goal of this problem is to study the power series $f(z) = \sum_{n=0}^{\infty} F_n z^n$, the generating function for the Fibonacci numbers.

   (a) Show that $(1 - z - z^2)f(z) = z$ as a formal power series and deduce $f(z) = \dfrac{z}{1 - z - z^2}$.

   (b) Find complex constants a, α, b, β such that $\dfrac{z}{1 - z - z^2} = \dfrac{a}{1 - \alpha z} + \dfrac{b}{1 - \beta z}$.

   (c) Prove Binet's formula for the Fibonacci numbers: $F_n = \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$ where $\varphi = \dfrac{1 + \sqrt{5}}{2}$ and $\overline{\varphi} = \dfrac{1 - \sqrt{5}}{2}$.
   [Hint: Expand the two geometric series from (b) and compare to $f(z)$.]

   (d) Find the radius of convergence of $f(z)$.

Remark: A similar method to the one in (a)-(c) can be used to solve any linear recurrence with constant coefficients, of the form $a_{n+1} = c_n a_n + \cdots + c_{n-k} a_{n-k}$ for constants c_i. Moreover, the general technique of considering the generating function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for an arbitrary sequence $a_0, a_1, \ldots$ can be used to find and prove many kinds of combinatorial identities.

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6. The goal of this problem is to prove that if p is any polynomial, then the formal power series $\sum_{n=0}^{\infty} p(n) z^n$ is a rational function in z.

   (a) Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that $zf'(z) = \sum_{n=0}^{\infty} n a_n z^n$.

   (b) Show that for every integer $k \geq 0$, $\sum_{n=0}^{\infty} n^k z^n$ is a rational function in z. [Hint: Induct on k.]

   (c) Show that $\sum_{n=0}^{\infty} p(n) z^n$ is a rational function in z for any polynomial $p(x) = b_d x^d + \cdots + b_0$.

   (d) Express $\sum_{n=0}^{\infty} (2n + 5) z^n$ and $\sum_{n=0}^{\infty} (n^2 + n) z^n$ as rational functions in z.

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7. [Challenge] The goal of this problem is to study the complex analogue of Newton's binomial series. Let α be any complex number that is not a nonnegative integer. Define the binomial coefficient $\dbinom{\alpha}{n} = \dfrac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - n + 1)}{n!}$ for each integer $n \geq 0$. Now define the binomial series $B_\alpha(z) = \sum_{n=0}^{\infty} \dbinom{\alpha}{n} z^n$. In 1665, Newton proved that if α is real, then $B_\alpha(x) = (1 + x)^\alpha$ for all real $-1 < x < 1$.

   (a) Show that the radius of convergence of the binomial series equals 1. [Hint: Use the Ratio Test.]

   (b) For a positive integer m, show that $(B_{1/m}(z))^m = 1 + z$ for all $|z| < 1$. [Hint: Use Newton's binomial theorem and the uniqueness of series expansions.]

   (c) Deduce that for $|z| < 1$, the binomial series $B_{1/m}(z)$ is a holomorphic function of z whose value is an mth root of $1 + z$.
