1. For f, g, h as below, perform the requested calculations:

$$f = \sum_{n=0}^{\infty} n^2 z^n = z + 4z^2 + 9z^3 + 16z^4 + 25z^5 + \cdots$$
$$g = \sum_{n=0}^{\infty} (1+2n) z^n = 1 + 3z + 5z^2 + 7z^3 + 9z^4 + 11z^5 + \cdots$$
$$h = \sum_{n=-2}^{\infty} (-1)^n z^n = z^{-2} - z^{-1} + 1 - z + z^2 - z^3 + z^4 - \cdots$$

(a) Find $f + 2g$ up through the terms of order 4.

- Add: $f+2g = (z+4z^2+9z^3+16z^4+\cdots)+2(1+3z+5z^2+7z^3+9z^4+\cdots) = \boxed{2 + 7z + 14z^2 + 23z^3 + 34z^4 + \cdots}$.

(b) Find $f + h$ up through the terms of order 4.

- Add: $f + h = (z + 4z^2 + 9z^3 + 16z^4 + \cdots) + (z^{-2} - z^{-1} + 1 - z + z^2 - z^3 + z^4 - \cdots) = \boxed{z^{-2} - z^{-1} + 1 + 5z^2 + 8z^3 + 17z^4 + \cdots}$.

(c) Find fg up through the terms of order 4.

- Multiply: $fg = (z+4z^2+9z^3+16z^4+\cdots)\cdot(1+3z+5z^2+7z^3+9z^4+\cdots) = \boxed{z + 7z^2 + 26z^3 + 70z^4 + \cdots}$.

(d) Find fh up through the terms of order 3.

- Multiply: $fh = (z + 4z^2 + 9z^3 + 16z^4 + \cdots) \cdot (z^{-2} - z^{-1} + 1 - z + z^2 - z^3 + z^4 - \cdots) = \boxed{z^{-1} + 3 + 6z + 10z^2 + 15z^3 + \cdots}$.

(e) Find f^2 up through the terms of order 5.

- Multiply: $f^2 = (z+4z^2+9z^3+16z^4+\cdots)\cdot(z+4z^2+9z^3+16z^4+\cdots) = \boxed{z^2 + 8z^3 + 34z^4 + 104z^5 + \cdots}$.

(f) Find $1/g$ up through the terms of order 4.

- If $g^{-1} = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \cdots$ then we get $1 = (1 + 3z + 5z^2 + 7z^3 + 9z^4 + \cdots)(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \cdots) = b_0 + (3b_0 + b_1)z + (5b_0 + 3b_1 + b_2)z^2 + (7b_0 + 5b_1 + 3b_2 + b_3)z^3 + (9b_0 + 7b_1 + 5b_2 + 3b_3 + b_4)z^4 + \cdots$.
- Setting coefficients equal yields $b_0 = 1$, $3b_0 + b_1 = 0$ so $b_1 = -3$, $5b_0 + 3b_1 + b_2 = 0$ so $b_2 = 4$, $7b_0 + 5b_1 + 3b_2 + b_3 = 0$ so $b_3 = -4$, $9b_0 + 7b_1 + 5b_2 + 3b_3 + b_4 = 0$ so $b_4 = 4$, and so forth.
- So $g^{-1} = \boxed{1 - 3z + 4z^2 - 4z^3 + 4z^4 + \cdots}$.

(g) Find $1/f$ up through the terms of order 3.

- Taking out the factor of z, we need to find the inverse of $f/z = 1 + 4z + 9z^2 + 16z^3 + 25z^4 + \cdots$.
- If the inverse is $b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \cdots$ then we get $1 = (1 + 4z + 9z^2 + 16z^3 + 25z^4 + \cdots)(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \cdots) = b_0 + (4b_0 + b_1)z + (9b_0 + 4b_1 + b_2)z^2 + (16b_0 + 9b_1 + 4b_2 + b_3)z^3 + (25b_0 + 16b_1 + 9b_2 + 4b_3 + b_4)z^4 + \cdots$.
- Setting coefficients equal yields $b_0 = 1$, $4b_0 + b_1 = 0$ so $b_1 = -4$, $9b_0 + 4b_1 + b_2 = 0$ so $b_2 = 7$, $16b_0 + 9b_1 + 4b_2 + b_3 = 0$ so $b_3 = -8$, $25b_0 + 16b_1 + 9b_2 + 4b_3 + b_4 = 0$ so $b_4 = 8$, and so forth.
- So $(f/z)^{-1} = (1-4z+7z^2-8z^3+8z^4+\cdots)$ and thus $f^{-1} = z^{-1}(f/z)^{-1} = \boxed{z^{-1} - 4 + 7z - 8z^2 + 8z^3 + \cdots}$.

(h) Find $1/h$. [Hint: It is a polynomial.]

- Taking out the factor of z^{-2} yields that we need to find the inverse of $z^2 h = 1 + z + z^3 + z^4 + z^5 + \cdots$, which as we have calculated is $1 - z$. Then $h^{-1} = z^2(z^2 h)^{-1} = z^2(1 - z) = \boxed{z^2 - z^3}$.

(i) Find f/g up through the terms of order 3.

- We simply evaluate $f \cdot g^{-1} = (z + 4z^2 + 9z^3 + 16z^4 + \cdots) \cdot (1 - 3z + 4z^2 - 4z^3 + 4z^4 + \cdots) = \boxed{z + z^2 + z^3 + z^4 + \cdots}$.

2. Find the radius of convergence for each power series:

   (a) $\sum_{n=0}^{\infty} \dfrac{2^n}{n+2} z^n$.

   - We compute $\lim_{n\to\infty} \left|\dfrac{2^n}{n+2}\right|^{1/n} = \lim_{n\to\infty} \dfrac{2}{(n+2)^{1/n}} = 2$. Thus, the radius of convergence is $\boxed{1/2}$.

   (b) $\sum_{n=1}^{\infty} \dfrac{z^n}{n^n}$.

   - We compute $\lim_{n\to\infty} \left|\dfrac{1}{n^n}\right|^{1/n} = \lim_{n\to\infty} \left|\dfrac{1}{n}\right| = 0$. Thus, the radius of convergence is $1/0 = \boxed{\infty}$.

   (c) $\sum_{n=1}^{\infty} n^n z^n$.

   - We compute $\lim_{n\to\infty} |n^n|^{1/n} = \lim_{n\to\infty} |n| = \infty$. Thus, the radius of convergence is $1/\infty = \boxed{0}$.

   (d) $\sum_{n=0}^{\infty} \dfrac{(3+2i)^n}{(1-i)^n} z^n$.

   - We compute $\lim_{n\to\infty} \left|\dfrac{(3+2i)^n}{(1-i)^n}\right|^{1/n} = \lim_{n\to\infty} \left|\dfrac{3+2i}{1-i}\right| = \sqrt{\dfrac{13}{2}}$. Thus, the radius of convergence is $\boxed{\sqrt{2/13}}$.

   (e) $\sum_{n=0}^{\infty} (i - \dfrac{1}{n})^n z^n$.

   - We compute $\lim_{n\to\infty} \left|(i-\dfrac{1}{n})^n\right|^{1/n} = \lim_{n\to\infty} \left|i - \dfrac{1}{n}\right| = 1$. Thus, the radius of convergence is $\boxed{1}$.

   (f) $\sum_{n=1}^{\infty} n^5 z^n$.

   - We compute $\lim_{n\to\infty} |n^5|^{1/n} = \lim_{n\to\infty} |n^{5/n}| = 1$. Thus, the radius of convergence is $1/1 = \boxed{1}$.

   (g) $\sum_{n=0}^{\infty} \dfrac{2^n}{3^n + 4^n} z^n$.

   - We compute $\lim_{n\to\infty} \left|\dfrac{2^n}{3^n+4^n}\right|^{1/n} = \lim_{n\to\infty} \dfrac{2}{4(1+(3/4)^n)^{1/n}} = \dfrac{1}{2}$. Thus, the radius of convergence is $\boxed{2}$.

   (h) $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} z^{2n+1}$.

   - We compute $\lim_{n\to\infty} \left|\dfrac{(-1)^n}{(2n+1)!}\right|^{1/n} = \lim_{n\to\infty} \left|\dfrac{1}{(2n+1)!}\right|^{1/n} = 0$, since $[(2n+1)!]^{1/n} > (2n/e)^2$. Thus, the radius of convergence is $1/0 = \boxed{\infty}$.

   (i) $\sum_{n=0}^{\infty} z^{2^n}$.

   - Note that the coefficients are all either 0 or 1. Although $\lim_{n\to\infty} |a_n|^{1/n}$ does not converge since it oscillates between 0 and 1, the limsup does converge to 1.
   - Thus, the radius of convergence is $\boxed{1}$.

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3. The goal of this problem is to prove the Ratio Test. So suppose $\{a_n\}_{n\geq 1}$ is a complex sequence whose terms are nonzero.

   (a) Suppose that there exists a real number $\rho < 1$ and positive N such that $|a_{n+1}/a_n| \leq \rho$ for all $n \geq N$. Show that $\sum_{n=1}^{\infty} a_n$ converges absolutely. [Hint: Explain why $|a_{N+k}| \leq |a_N| \rho^k$ for all $k \geq 0$ and use this to show that $\sum_{n=N}^{\infty} |a_n|$ is finite.]

   - Suppose $\rho < 1$ and that $|a_{n+1}/a_n| \leq \rho$ for all $n \geq N$. Then by multiplying the inequalities for $n = N, N+1, \ldots, N+k-1$ we see that $|a_{N+k}| \leq |a_N| \rho^k$ for all $k \geq 0$.

- Then $\sum_{n=N}^{\infty} |a_n| \le \sum_{n=N}^{\infty} |a_N| \rho^n = \dfrac{|a_N| \rho^n}{1-\rho}$. This means $\sum_{n=N}^{\infty} |a_n|$ converges, and therefore so does $\sum_{n=1}^{\infty} |a_n|$, since it only has a finite number of additional terms.
- This means $\sum_{n=1}^{\infty} a_n$ converges absolutely as claimed.

(b) Give an example of a series with $|a_{n+1}/a_n| < 1$ for all n but where Show that if we weaken the hypothesis of (a) only to require that $|a_{n+1}/a_n| < 1$ for all n, then $\sum_{n=1}^{\infty} a_n$ need not converge absolutely.

- A simple counterexample is to take $a_n = \dfrac{1}{n}$: then $a_{n+1} < a_n$ for all n, but $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \dfrac{1}{n}$ does not converge absolutely (or at all).
- The issue is that even if $|a_{n+1}/a_n| < 1$, we can still have $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, in which case we don't get the necessary exponential decay in the sizes of the terms.

(c) Suppose that there exists a positive N such that $|a_{n+1}/a_n| \ge 1$ for all $n \ge N$. Show that $\sum_{n=1}^{\infty} a_n$ diverges. [Hint: The terms cannot go to zero.]

- Suppose that $|a_{n+1}/a_n| \ge 1$ for all $n \ge N$. Then by a trivial induction we have $|a_{N+k}| \ge |a_N|$ for all $k \ge 0$.
- Then the terms of the series do not tend to zero, since their absolute values are all at least $|a_N| > 0$, so the series diverges.

(d) Prove the <u>Ratio Test</u>: If $\lim_{n\to\infty} |a_{n+1}/a_n| = r$ exists, then $\sum_{n=1}^{\infty} a_n$ converges absolutely if $r < 1$ and $\sum_{n=1}^{\infty} a_n$ diverges if $r > 1$. [Hint: For the convergence, take any ρ with $r < \rho < 1$ in (a).]

- Suppose that $\lim_{n\to\infty} |a_{n+1}/a_n| = \rho$.
- If $r < 1$ then take any ρ with $r < \rho < 1$. Then by the definition of limit there exists N such that for all $n \ge N$ we have $|a_{n+1}/a_n| < \rho$, so using this ρ in (a) immediately yields that $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $r > 1$ then similarly by the definition of limit there exists N such that for all $n \ge N$ we have $|a_{n+1}/a_n| > 1$, in which case (c) immediately yields that $\sum_{n=1}^{\infty} a_n$ diverges.

(e) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \dfrac{n^n}{n!} z^n$.

- We try using the Ratio Test to analyze the ratios between consecutive terms. With $a_n = \dfrac{n^n}{n!} z^n$, we see $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{n+1} z^{n+1}/(n+1)!}{n^n z^n / n!} = \dfrac{(n+1)^n}{n^n} z = (1+\dfrac{1}{n})^n z$.
- Thus, $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} (1+\dfrac{1}{n})^n |z| = e|z|$ by the usual limit $\lim_{n\to\infty} (1+\dfrac{1}{n})^n = e$.
- So by the Ratio Test, if $|z| < 1/e$ we see that $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| < 1$ and so the power series converges, while if $|z| > 1/e$ then $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| > 1$ so the series diverges.
- Thus, the radius of convergence must be $\boxed{1/e}$.

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4. Suppose $f = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence R_f and $g = \sum_{n=0}^{\infty} b_n z^n$ is a power series with radius of convergence R_g, where $R_f \le R_g$.

(a) Show that $f+g$ has radius of convergence at least R_f.

- As shown in class, $f(z)$ converges absolutely for $|z| < R_f$ and $g(z)$ converges absolutely for $|z| < R_g$ hence in particular for $|z| < R_f$.
- Thus the sum $f(z) + g(z)$ also converges absolutely for $|z| < R_f$, meaning the radius of convergence is at least R_f.

(b) If $R_f < R_g$ show that $f+g$ has radius of convergence exactly R_f.

- By (a) we just need to show that the radius of convergence cannot be larger than R_f.

- Since $R_f < R_g$, suppose z has $R_f < |z| < R_g$. Then $f(z)$ diverges (as shown in class) but $g(z)$ converges.
- Then $f(z) + g(z)$ must also diverge, since if it converged, so would $[f(z) + g(z)] - g(z) = f(z)$.
- This means $f + g$ diverges whenever $|z| > R_f$, and so the radius of convergence of $f + g$ cannot be larger than R_f.

(c) Find an example of power series f and g with $R_f = R_g$ such that $f + g$ has radius of convergence strictly greater than R_f.

- A simple example is to take $g = -f$ where f has a finite radius of convergence (e.g., $f = \sum_{n=0}^{\infty} z^n$ with $R_f = 1$): then g has the same radius of convergence as f, but $f + g = 0$ has radius of convergence ∞.
- In fact, in any possible example we must have $g = -f + h$ where h has radius of convergence bigger than f. But in fact, all such examples actually work since by (b) g and f have the same radius of convergence while $f + g = h$ has a larger radius of convergence.

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5. [Challenge] The goal of this problem is to discuss a useful summation technique known as Abel summation (also called summation by parts), and then use it to derive some series convergence tests. So suppose $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are two complex sequences, and define $S_n = \sum_{k=1}^{n} a_k b_k$ and $B_n = \sum_{k=1}^{n} b_k$.

(a) Show the Abel summation formula: that $S_n = a_n B_n + \sum_{k=1}^{n-1} B_k(a_k - a_{k+1})$.

- We induct on n. For the base case $n = 1$ we have $S_1 = a_1 B_1 + 0$ as required.
- For the inductive step suppose that $S_n = a_n B_n + \sum_{k=1}^{n-1} B_k(a_k - a_{k+1})$.
- Then

$$\begin{aligned} S_{n+1} &= a_{n+1}b_{n+1} + S_n \\ &= a_{n+1}b_{n+1} + [a_n B_n + \sum_{k=1}^{n-1} B_k(a_k - a_{k+1})] \\ &= a_{n+1}B_{n+1} + (a_n - a_{n+1})B_n + \sum_{k=1}^{n-1} B_k(a_k - a_{k+1}) \\ &= a_{n+1}B_{n+1} + \sum_{k=1}^{n} B_k(a_k - a_{k+1}) \end{aligned}$$

as required.

(b) Prove Dirichlet's convergence test: if $\{a_n\}_{n\geq 1}$ is a strictly decreasing sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$ and the sequence $\{B_n\}_{n\geq 1}$ is bounded, then $\sum_{k=1}^{\infty} a_k b_k$ converges. [Hint: Suppose $|B_n| \leq M$ for all n. Use (a) on the partial sum S_n and then show that $a_n B_n \to 0$ and that $\sum_{k=1}^{n-1} B_k(a_k - a_{k+1})$ is absolutely convergent.]

- Suppose that $\{a_n\}_{n\geq 1}$ is a strictly decreasing sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$ and the sequence $\{B_n\}_{n\geq 1}$ is bounded, say with $|B_n| \leq M$ for all n.
- Following the hint, applying the Abel summation formula from (a) yields $S_n = a_n B_n + \sum_{k=1}^{n-1} B_k(a_k - a_{k+1})$.
- Since $|B_n| \leq M$ for all n, we have $|a_n B_n| \leq M a_n \to 0$ as $n \to \infty$ since $a_n \to 0$ by hypothesis.
- Furthermore, because $a_k > a_{k+1}$ we have $|a_k - a_{k+1}| = a_k - a_{k+1}$.
- Therefore $\sum_{k=1}^{n-1} |B_k(a_k - a_{k+1})| = \sum_{k=1}^{n-1}(a_k - a_{k+1}) |B_k| \leq \sum_{k=1}^{n-1}(a_k - a_{k+1})M = M(a_1 - a_n)$, so taking the limit as $n \to \infty$ yields $M a_1$, which is finite. This means $\sum_{k=1}^{\infty} B_k(a_k - a_{k+1})$ converges absolutely, hence converges.
- Putting this together we see that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} a_n B_n + \sum_{k=1}^{\infty} B_k(a_k - a_{k+1})$ converges, as required.

(c) Deduce the Alternating Series Test: if $\{a_n\}_{n\geq 1}$ is a strictly decreasing sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

- This follows immediately from Dirichlet's convergence test applied with $b_k = (-1)^k$: clearly the sums $B_n = \sum_{k=1}^n (-1)^k$ are bounded since they alternate between -1 and 0, and so all of the hypotheses are satisfied.
- We conclude that $\sum_{k=1}^\infty a_k b_k = \sum_{k=1}^\infty (-1)^k a_k$ converges, as desired.

• Remark: The summation formula in part (a) is the discrete analogue of integration by parts, whence its name "summation by parts". Specifically, the sum $\sum_{k=1}^n f(k)$ is the analogue of the antiderivative $\int f(x)\,dx$ while the difference $f(k+1) - f(k)$ is the analogue of the derivative $f'(x)$. The formula in (a) is then the analogue of the integration by parts formula $\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$ where G is an antiderivative of g (the minus sign arises because the differences in (a) are $a_k - a_{k+1}$ rather than $a_{k+1} - a_k$).

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6. The goal of this problem is to study the convergence of the power series $f = \sum_{n=1}^\infty \frac{1}{n} z^n$.

(a) Show that f has radius of convergence 1 and explain why f does not converge for $z = 1$.

- We have $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} n^{-1/n} = 1$ so the radius of convergence is 1 as claimed.
- Additionally, if $z = 1$ then the series becomes $\sum_{n=1}^\infty \frac{1}{n}$, which diverges: it is the usual harmonic series, which goes to ∞ by the integral estimate $\sum_{n=1}^t \frac{1}{n} > \int_1^t \frac{1}{t}\,dt = \ln(t)$.

(b) Show that if $0 < \theta < 2\pi$, then $\left|\sum_{k=0}^n e^{ik\theta}\right| \le \frac{2}{|e^{i\theta} - 1|}$. [Hint: Use problem 6(d) of homework 1.]

- In problem 6(d) of homework 1, we showed that $1 + e^{ix} + e^{2ix} + \cdots + e^{inx} = \frac{e^{(n+1)ix} - 1}{e^{ix} - 1}$.
- Taking the absolute value and writing θ in place of x yields $\left|\sum_{k=0}^n e^{i\theta}\right| = \frac{\left|e^{(n+1)i\theta} - 1\right|}{|e^{i\theta} - 1|}$.
- Since $\frac{\left|e^{(n+1)i\theta} - 1\right|}{|e^{i\theta} - 1|} \le \frac{\left|e^{(n+1)i\theta}\right| + |-1|}{|e^{i\theta} - 1|} = \frac{2}{|e^{i\theta} - 1|}$ by the triangle inequality, the result follows immediately.

(c) Show that f does converge if $|z| = 1$ but $z \ne 1$. [Hint: Use (b) with $z = e^{i\theta}$ in Dirichlet's convergence test from problem 5(b); you just need to verify all of the hypotheses hold.]

- Suppose $|z| = 1$ so that $z = e^{i\theta}$. Then $z \ne 1$ allows us to assume $0 < \theta < 2\pi$.
- Per the hint we will apply Dirichlet's convergence test with $a_n = \frac{1}{n}$ and $b_n = z^n = e^{ni\theta}$, so we need to check the hypotheses.
- Clearly $\{a_n\}_{n\ge1}$ is a strictly decreasing sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$.
- Additionally, by part (b), the sums $B_n = \sum_{k=0}^n z^n = \sum_{k=0}^n e^{ki\theta}$ are all bounded in absolute value by $\frac{2}{|e^{i\theta} - 1|}$.
- Therefore, by Dirichlet's convergence test, the sum $\sum_{n=1}^\infty a_n b_n = \sum_{n=1}^\infty \frac{1}{n} z^n$ converges, as claimed.
- Combining with the result of (a) we deduce that f converges absolutely for all $|z| < 1$ and also when $|z| = 1$ except at the point $z = 1$ where it diverges.

(d) Determine all z for which $f = \sum_{n=0}^\infty \frac{z^n}{n}$ converges.

- From (a) since the radius of convergence is 1 we know that f converges for all $|z| < 1$ and diverges for all $|z| > 1$. Also from (a) we know f diverges for $z = 1$.
- From (c) we know f converges for all $|z| = 1$ with $z \ne 1$.
- Thus, f converges for all $|z| \le 1$ except for $z = 1$, and it diverges for all $|z| > 1$ along with $z = 1$.
