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1. For each complex function, calculate its partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$, and determine whether the complex derivative f' exists on any open region R.

   (a) $f(z) = z^4 + z$.

   - We have $\frac{\partial f}{\partial z} = \boxed{4z^3 + 1}$ and $\frac{\partial f}{\partial \overline{z}} = \boxed{0}$. The complex derivative f' exists everywhere since $\frac{\partial f}{\partial \overline{z}} = 0$.

   (b) $f(z) = \overline{z^4 + z}$.

   - We have $\frac{\partial f}{\partial z} = \boxed{0}$ and $\frac{\partial f}{\partial \overline{z}} = \boxed{4\overline{z}^3 + \overline{z}}$. The complex derivative f' does not exist except possibly when $4\overline{z}^3 + \overline{z} = 0$, which only occurs for three values of z (namely, $z = 0, \pm i/2$). Any open region contains more than these points, so f' does not exist on any open region.

   (c) $f(z) = 3z\overline{z}^2 + z^4$.

   - We have $\frac{\partial f}{\partial z} = \boxed{3\overline{z}^2 + 4z^3}$ and $\frac{\partial f}{\partial \overline{z}} = \boxed{6z\overline{z}}$. The complex derivative f' does not exist except possibly when $z = 0$. Since this is just a single point, f' does not exist on any open region.

   (d) $f(z) = \frac{e^z}{\overline{z} - 1}$.

   - We have $\frac{\partial f}{\partial z} = \boxed{\frac{e^z}{\overline{z} - 1}}$ and $\frac{\partial f}{\partial \overline{z}} = \boxed{-\frac{e^z}{(\overline{z} - 1)^2}}$. The complex derivative f' does not exist anywhere since $\frac{\partial f}{\partial \overline{z}}$ is never zero.

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2. For each complex function, calculate its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and determine whether the complex derivative f' exists using the Cauchy-Riemann equations.

   (a) $f(x + iy) = (2x^2 + y) + (2y^2 - x)i$.

   - We have $\frac{\partial f}{\partial x} = \boxed{4x - i}$ and $\frac{\partial f}{\partial y} = \boxed{1 + 4yi}$. Since $\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right]$ is not zero, we see f' $\boxed{\text{does not exist}}$.

   (b) $f(x + iy) = 4xy + (2y^2 - 2x^2)i$.

   - We have $\frac{\partial f}{\partial x} = \boxed{4y - 4xi}$ and $\frac{\partial f}{\partial y} = \boxed{4x + 4yi}$. Since $\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[(4y - 4xi) + i(4x + 4yi)\right] = 0$, we see f' $\boxed{\text{does exist}}$ everywhere.

   (c) $f(x + iy) = (3 + e^y \sin x) - (e^y \cos x)i$.

   - We have $\frac{\partial f}{\partial x} = \boxed{e^y \cos x + ie^y \sin x}$ and $\frac{\partial f}{\partial y} = \boxed{e^y \sin x - ie^y \cos x}$. Since $\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[(e^y \cos x + ie^y \sin x) + i(e^y \sin x\right.$ is not zero, we see f' $\boxed{\text{does not exist}}$. (In fact, the imaginary term is off by a factor of -1.)

   (d) $f(x + iy) = \sin x \cos y - i \cos x \sin y$.

   - We have $\frac{\partial f}{\partial x} = \boxed{\cos x \cos y + i \sin x \sin y}$ and $\frac{\partial f}{\partial y} = \boxed{-\sin x \sin y - i \cos x \cos y}$. Since $\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[(\cos x \cos y + i \sin x \sin y) + i(-\sin x \sin y - i \cos x \cos y)\right] = 2 \cos x \cos y$ is not zero, we see f' $\boxed{\text{does not exist}}$.

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3. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is twice differentiable. We define the Laplacian of f to be $\Delta f = \nabla^2 \cdot f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f_{xx} + f_{yy}$, and we say f is harmonic if $\Delta f = 0$ on the entire domain of f.

(a) Find the Laplacians of $3x - y$, $x^2 - y^2$, e^{x+y}, $e^x \cos y$, $e^y \cos x$, $\frac{1}{x^2 + y^2}$, $\ln(x^2 + y^2)$, and $\tan^{-1}(y/x)$. Which of these are harmonic?

- $\Delta(3x - y) = 0 + 0 = 0$.
- $\Delta(x^2 - y^2) = 2 - 2 = 0$.
- $\Delta(e^{x+y}) = e^{x+y} + e^{x+y} = 2e^{x+y}$.
- $\Delta(e^x \cos y) = e^x \cos y - e^x \cos y = 0$.
- $\Delta(e^y \cos x) = -e^y \cos x + e^y \cos x = 0$.
- $\Delta(\frac{1}{x^2 + y^2}) = \frac{6x^2 - 2y^2}{(x^2 + y^2)^3} + \frac{6y^2 - 2x^2}{(x^2 + y^2)^3} = \frac{4}{(x^2 + y^2)^2}$.
- $\Delta(\ln(x^2 + y^2)) = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} = 0$.
- $\Delta(\tan^{-1}(y/x)) = \frac{2xy}{(x^2 + y^2)^2} + \frac{-2xy}{(x^2 + y^2)^2} = 0$.
- We see that $3x - y$, $x^2 - y^2$, $e^x \cos y$, $e^y \cos x$, $\ln(x^2 + y^2)$, and $\tan^{-1}(y/x)$ are harmonic.

(b) Suppose $h(z) = f(x, y) + ig(x, y)$ is a function of $z = x + iy$ where f and g are both twice continuously differentiable. Show that $4\frac{\partial^2 f}{\partial z \partial \bar{z}} = \Delta f$. [Hint: Partial derivatives can be interchanged for twice continuously differentiable functions.]

- As noted in the hint, since f and g are both twice continuously differentiable, we may arbitrarily interchange the order of partial derivatives.
- Then $4\frac{\partial^2}{\partial z \partial \bar{z}} = 4 \cdot \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] \cdot \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] = \frac{\partial^2}{\partial x^2} - i\frac{\partial^2}{\partial y \partial x} + i\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \Delta$.

(c) Suppose $h(z) = f(x, y) + ig(x, y)$ is a holomorphic function of $z = x + iy$ on the region R. Show that f and g are harmonic on R.

- Since h is holomorphic we have the Cauchy-Riemann equations $\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$ and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.
- Taking the x-derivative of the first and the y-derivative of the second and adding yields $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\frac{\partial^2 g}{\partial y \partial x} + \frac{\partial^2 g}{\partial x \partial y} = 0$ by interchanging the order of the derivatives (which we may do because f, g are C^2 because h is holomorphic). Thus f is harmonic.
- Similarly, taking the x-derivative of the second and subtracting the y-derivative of the first yields $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = -\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y} = 0$ again by interchanging the order of the derivatives. Thus g is also harmonic.

Remark: Part (c) shows that the real and imaginary parts of a holomorphic function are harmonic. As we will show later in the semester, the converse is also broadly true: a harmonic function defined on a sufficiently nice region is necessarily the real (or imaginary) part of a holomorphic function.

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4. Suppose that we define two differential operators $L = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ and $M = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}$ for some constants $a, b, c, d \in \mathbb{C}$, meaning that $Lf = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$ and similarly $Mf = c\frac{\partial f}{\partial x} + d\frac{\partial f}{\partial y}$ for a function f. Show that if $Lz = 1$, $L\overline{z} = 0$, $Mz = 0$, and $M\overline{z} = 1$ for all $z, \overline{z}$, then in fact we must have $a = \frac{1}{2}$, $b = -\frac{i}{2}$, $c = \frac{1}{2}$, $d = \frac{i}{2}$ so that $L = \frac{\partial}{\partial z}$ and $M = \frac{\partial}{\partial \overline{z}}$.

- Since $z = x + iy$ we have $\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial y} = i$, so $Lz = a + bi$ and $Mz = c + di$.
- Likewise, since $\overline{z} = x - iy$ we have $\frac{\partial \overline{z}}{\partial x} = 1$ and $\frac{\partial \overline{z}}{\partial y} = -i$, so $L\overline{z} = a - bi$ and $M\overline{z} = c - di$.
- So the given conditions yield $a + bi = 1$, $a - bi = 0$, $c + di = 0$, $c - di = 1$. Solving this easy system yields $a = \frac{1}{2}$, $b = -\frac{i}{2}$, $c = \frac{1}{2}$, $d = \frac{i}{2}$, and thus $L = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] = \frac{\partial}{\partial z}$ and $M = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] = \frac{\partial}{\partial \overline{z}}$, as claimed.

Remark: The point of this calculation is that our definitions of $L = \frac{\partial}{\partial z}$ and $M = \frac{\partial}{\partial \overline{z}}$ are forced to be the ones we selected if we want them to act on z and $\overline{z}$ in the expected way.

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5. We have already discussed how to convert between the "rectangular" differential operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and the "complex" differential operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}$. The goal of this problem is to write down the "polar" differential operators $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$.

(a) Suppose $f(z)$ is differentiable where $z = re^{i\theta}$. Show that $\frac{\partial f}{\partial r} = \frac{1}{r}\left[z\frac{\partial f}{\partial z} + \overline{z}\frac{\partial f}{\partial \overline{z}}\right]$ and $\frac{\partial f}{\partial \theta} = i\left[z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}}\right]$. [Hint: Note that $\overline{z} = re^{-i\theta}$ and then use the chain rule.]

- Noting that $z = re^{i\theta}$ and $\overline{z} = re^{-i\theta}$, differentiating yields $\frac{\partial z}{\partial r} = e^{i\theta}$, $\frac{\partial z}{\partial \theta} = ire^{i\theta}$, $\frac{\partial \overline{z}}{\partial r} = e^{-i\theta}$, and $\frac{\partial \overline{z}}{\partial \theta} = -ire^{i\theta}$.
- Then by the multivariable chain rule we have $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r} + \frac{\partial f}{\partial \overline{z}} \cdot \frac{\partial \overline{z}}{\partial r} = e^{i\theta}\frac{\partial f}{\partial z} + e^{-i\theta}\frac{\partial f}{\partial \overline{z}} = \frac{1}{r}\left[z\frac{\partial f}{\partial z} + \overline{z}\frac{\partial f}{\partial \overline{z}}\right]$ as claimed.
- Likewise, $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \theta} + \frac{\partial f}{\partial \overline{z}} \cdot \frac{\partial \overline{z}}{\partial \theta} = ire^{i\theta}\frac{\partial f}{\partial z} - ire^{-i\theta}\frac{\partial f}{\partial \overline{z}} = i\left[z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}}\right]$.

(b) Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ for $f(z) = z^2$ and for $f(z) = z^3\overline{z}^3$. Do these agree with the expected expressions for $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ when f is written in terms of r and θ?

- For $f(z) = z^2$ we see $\frac{\partial f}{\partial r} = \frac{1}{r}[z \cdot 2z + \overline{z} \cdot 0] = \frac{2z^2}{r} = 2re^{2i\theta}$ and $\frac{\partial f}{\partial \theta} = i\left[z \cdot 2z - \overline{z} \cdot 0\right] = 2iz^2 = 2ir^2e^{2i\theta}$.
- If we write f in terms of r and θ we have $f(z) = r^2e^{2i\theta}$ which would lead to partial derivatives $\frac{\partial f}{\partial r} = 2re^{2i\theta}$ and $\frac{\partial f}{\partial \theta} = 2ir^2e^{2i\theta}$, which is exactly as obtained.
- For $f(z) = z^3\overline{z}^3$ we see $\frac{\partial f}{\partial r} = \frac{1}{r}[z \cdot 3z^2\overline{z}^3 + \overline{z} \cdot 3z^3\overline{z}^2] = \frac{6z^3\overline{z}^3}{r} = 6r^5$ and $\frac{\partial f}{\partial \theta} = i\left[z \cdot 3z^2\overline{z}^3 - \overline{z} \cdot 3z^3\overline{z}^2\right] = 0$.
- If we write f in terms of r and θ we have $f(z) = r^6$ which would lead to partial derivatives $\frac{\partial f}{\partial r} = 6r^5$ and $\frac{\partial f}{\partial \theta} = 0$, which is exactly as obtained.

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6. Recall that for a positive integer n, the <u>nth roots of unity</u> are the solutions to the equation $z^n = 1$, and are given explicitly by $\{1, e^{2\pi i/n}, e^{4\pi i/n}, \ldots, e^{2(n-1)\pi i/n}\} = \{1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}\}$ where $\zeta_n = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ for shorthand. The goal of this problem is to explore some results about these numbers.

   (a) Give, in explicit $a + bi$ form, the 3rd, 6th, and 8th roots of unity.

   - The 3rd roots: $\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}$ for $k = 0, 1, 2$: $\boxed{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i}$.
   - The 6th roots: $\cos\frac{2\pi k}{6} + i\sin\frac{2\pi k}{6}$ for $k = 0, 1, \ldots, 5$: $\boxed{1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i}$.
   - The 8th roots: $\cos\frac{2\pi k}{8} + i\sin\frac{2\pi k}{8}$ for $k = 0, 1, \ldots, 7$: $\boxed{1, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -1, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, -i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i}$.

   (b) Show that $1/\zeta_n = \zeta_n^{n-1} = \overline{\zeta_n}$.

   - We have $\zeta_n^n = 1$ so dividing by ζ_n yields $\zeta_n^{n-1} = 1/\zeta_n$.
   - But also since $|\zeta_n| = 1$ squaring yields $|\zeta_n|^2 = \zeta_n\overline{\zeta_n} = 1$ so $\overline{\zeta_n} = 1/\zeta_n$ also.

   (c) Show that $\zeta_n^{n-1} + \zeta_n^{n-2} + \cdots + \zeta_n = -1$. [Hint: What are the roots of $p(z) = (z^n - 1)/(z - 1)$?]

   - Per the hint we observe that ζ_n is a root of the polynomial $p(z) = (z^n - 1)/(z - 1) = z^{n-1} + z^{n-2} + \cdots + z + 1$, since it makes the numerator zero but not the denominator.
   - Thus, $p(\zeta_n) = 0$, which is to say, $\zeta_n^{n-1} + \zeta_n^{n-2} + \cdots + \zeta_n + 1 = 0$ so that $\zeta_n^{n-1} + \zeta_n^{n-2} + \cdots + \zeta_n = -1$.

   (d) Let $\alpha = \zeta_5 + \zeta_5^4$ and $\beta = \zeta_5^2 + \zeta_5^3$. Show that $\alpha + \beta = -1$ and $\alpha\beta = -1$ and deduce that α and β are the roots of the quadratic $p(z) = z^2 + z - 1$. Use this along with the fact that $\alpha > 0$ to find α and β explicitly.

   - We have $\alpha + \beta = \zeta_5 + \zeta_5^4 + \zeta_5^2 + \zeta_5^3 = \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 = -1$ by the identity in (c).
   - Likewise, $\alpha\beta = (\zeta_5 + \zeta_5^4)(\zeta_5^2 + \zeta_5^3) = \zeta_5^3 + \zeta_5^4 + \zeta_5^6 + \zeta_5^7 = \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 = -1$ again by (c) and the fact that $\zeta_5^6 = \zeta_5$ and $\zeta_5^7 = \zeta_5^2$.
   - Then for $p(z) = (z - \alpha)(z - \beta) = z^2 - (\alpha + \beta)z + \alpha\beta$ plugging these values in shows $p(z) = z^2 + z - 1$.
   - By the quadratic formula, the roots of $p(z)$ are $\dfrac{-1 \pm \sqrt{5}}{2}$, and so α, β must equal these values in some order. Since $\alpha > 0$, that means $\boxed{\alpha = \dfrac{-1 + \sqrt{5}}{2}}$ and $\boxed{\beta = \dfrac{-1 - \sqrt{5}}{2}}$.

   (e) Show that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ and use this to give an explicit formula for the 5th root of unity ζ_5. [Hint: Use (d) and $\overline{\zeta_5} = \zeta_5^4$.]

   - By part 7(b) we have $\overline{\zeta_5} = \zeta_5^4$ so then (d) gives $\dfrac{-1 + \sqrt{5}}{2} = \alpha = \zeta_5 + \zeta_5^4 = \zeta_5 + \overline{\zeta_5} = 2\text{Re}(\zeta_5) = 2\cos\frac{2\pi}{5}$ and so $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ as claimed.
   - Then $\sin\frac{2\pi}{5} = \sqrt{1 - \cos^2\frac{2\pi}{5}} = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$, and finally $\zeta_5 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \boxed{\frac{\sqrt{5}-1}{4} + \frac{\sqrt{10+2\sqrt{5}}}{4}i}$.

   (f) Prove that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2}$ and that $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{1}{2}\sqrt{7}$. [Hint: Let $\alpha = \zeta_7 + \zeta_7^2 + \zeta_7^4$ and $\beta = \zeta_7^3 + \zeta_7^5 + \zeta_7^6$ and use a method similar to (d).]

   - Let $\alpha = \zeta_7 + \zeta_7^2 + \zeta_7^4$ and $\beta = \zeta_7^6 + \zeta_7^5 + \zeta_7^3$. Then $\alpha + \beta = \zeta_7 + \zeta_7^2 + \zeta_7^3 + \zeta_7^4 + \zeta_7^5 + \zeta_7^6 = -1$ and $\alpha\overline{\alpha} = (\zeta_7 + \zeta_7^2 + \zeta_7^4)(\zeta_7^6 + \zeta_7^5 + \zeta_7^3) = \zeta_7^4 + \zeta_7^5 + \zeta_7^6 + 3\zeta_7^7 + \zeta_7^8 + \zeta_7^9 + \zeta_7^{10} = 2$ using the identity $1 + \zeta_7 + \zeta_7^2 + \zeta_7^3 + \zeta_7^4 + \zeta_7^5 + \zeta_7^6 = 0$ from part (b).
   - Thus α and β are the two roots of the polynomial $p(z) = z^2 + z + 2$, so by the quadratic formula we see $\alpha, \beta = \dfrac{-1 \pm i\sqrt{7}}{2}$. Since the imaginary part of α is clearly positive, we have $\alpha = \dfrac{-1 + i\sqrt{7}}{2}$. Since $\text{Re}(\alpha) = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}$ and $\text{Im}(\alpha) = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$, setting the real and imaginary parts equal yields the claimed identities.

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7. Suppose z and w are complex numbers. The goal of this problem is to study some properties of the rational function $f_w(z) = \dfrac{z-w}{1-z\overline{w}}$, which when $|w| < 1$ is called a Möbius transformation.

   (a) Show that $f_w(z) = \dfrac{z-w}{1-z\overline{w}}$ is an invertible function with inverse function $f_{-w}(z) = \dfrac{z+w}{1+z\overline{w}}$.

   - Note that $f_w(f_{-w}(z)) = f_w(\dfrac{z+w}{1-z\overline{w}}) = \dfrac{\dfrac{z+w}{1-z\overline{w}} - w}{1 - \dfrac{z+w}{1-z\overline{w}}\overline{w}} = \dfrac{z+w-w(1-z\overline{w})}{(1-z\overline{w})-(z+w)\overline{w}} = \dfrac{z(1+w\overline{w})}{1+w\overline{w}} = z$ and likewise $f_{-w}(f_w(z)) = z$ by a similar calculation, so f is invertible with inverse as claimed.
   - Alternatively, solving $z' = \dfrac{z-w}{1-z\overline{w}}$ for z yields $z - w = z' - zz'\overline{w}$ so $z(1+z'\overline{w}) = z'+w$ so $z = \dfrac{z'+w}{1+z'\overline{w}}$. This shows $f_w^{-1}(z) = \dfrac{z+w}{1+z\overline{w}} = f_{-w}(z)$ once again.

   (b) If $z\overline{w} \neq 1$, show that $1 - \left|\dfrac{z-w}{1-z\overline{w}}\right|^2 = \dfrac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}$.

   - We have $1 - \left|\dfrac{z-w}{1-z\overline{w}}\right|^2 = 1 - \dfrac{z-w}{1-z\overline{w}} \cdot \dfrac{\overline{z}-\overline{w}}{1-\overline{z}w} = 1 - \dfrac{z\overline{z} - w\overline{z} - \overline{w}z + w\overline{w}}{1 - w\overline{z} - \overline{w}z + z\overline{z}w\overline{w}} = \dfrac{1 - z\overline{z} - w\overline{w} + z\overline{z}w\overline{w}}{(1-z\overline{w})(1-\overline{z}w)} = \dfrac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}$ as required.

   (c) Suppose that $|w| < 1$. Show that if $|z| = 1$ then $|f_w(z)| = |f_{-w}(z)| = 1$ and deduce that f_w is a bijection from the unit circle $|z| = 1$ to itself.

   - Suppose that $|z| = 1$. Then since $|w| < 1$ we see $z\overline{w} \neq 1$, so by part (b) we have $1 - |f_w(z)|^2 = \dfrac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2} = 0$, and thus $|f_w(z)| = 1$. By the same calculation we see that $|f_{-w}(z)| = 1$ also.
   - By (a) since f_w and f_{-w} are inverses, this means $|z| = 1$ is equivalent to $|f_w(z)| = 1$ and so f is a bijection from the unit circle $|z| = 1$ to itself.

   (d) Suppose that $|w| < 1$. Show that if $|z| < 1$ then $|f_w(z)| < 1$ and $|f_{-w}(z)| < 1$ and deduce that f_w is a bijection on the interior of the unit disc $|z| < 1$.

   - Similarly to (c), if $|w| < 1$ and $|z| < 1$ then $|z\overline{w}| < 1$ so $z\overline{w} \neq 0$ hence $|1 - z\overline{w}| > 0$.
   - Then by (b), we have $1 - |f_w(z)|^2 = \dfrac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2} > 0$ since the numerator and denominator are both positive. This means $|f_w(z)| < 1$, so f maps the interior of the unit disc to itself. Since $f_w^{-1}(z)$ has the same form, we see that $\left|f^{-1}(z)\right| < 1$ as well and so f^{-1} also maps the interior of the unit disc to itself.
   - Putting these two together shows that f is a bijection on the interior of the unit disc.

Remark: Parts (c) and (d) show that the function f maps the interior of the unit disc to itself, and also maps the disc's boundary to itself. As we will see, this phenomenon of mapping open regions to open regions, and boundaries to boundaries, are both general properties of holomorphic functions.

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8. [Challenge] Suppose that $p(z) = a(z-r_1)(z-r_2)\cdots(z-r_n)$ is a complex polynomial with roots $r_1, \dots r_n \in \mathbb{C}$. The goal of this problem is to prove the Gauss-Lucas theorem: that the roots of the derivative $p'(z)$ all lie inside of the smallest convex polygon that contains all of the roots $r_1, \dots, r_n$ of $p(z)$.

   (a) Show that $\dfrac{p'(z)}{p(z)} = \dfrac{1}{z-r_1} + \dfrac{1}{z-r_2} + \cdots + \dfrac{1}{z-r_n}$.

   - By the product rule we have $p'(z) = a \cdot 1 \cdot (z-r_2)(z-r_3)\cdots(z-r_n) + a(z-r_1)\cdot 1 \cdot (z-r_3)\cdots(z-r_n) + \cdots + a(z-r_1)(z-r_2)\cdots(z-r_{n-1}) \cdot 1$.
   - Each product is simply $p(z)$ with one of the terms $z - r_i$ replaced with 1, so dividing yields $\dfrac{p'(z)}{p(z)} = \dfrac{1}{z-r_1} + \dfrac{1}{z-r_2} + \cdots + \dfrac{1}{z-r_n}$.

   (b) Suppose that the imaginary parts of $r_1, \dots, r_n$ are positive while the imaginary part of z_0 is negative. Show that $p'(z_0) \neq 0$.

   - Suppose that the imaginary parts of $r_1, \dots, r_n$ are positive while the imaginary part of z_0 is negative.
   - Then from $\dfrac{p'(z_0)}{p(z_0)} = \dfrac{1}{z_0-r_1} + \dfrac{1}{z_0-r_2} + \cdots + \dfrac{1}{z_0-r_n}$, we see that each of the terms $z_0 - r_i$ has a negative imaginary part. Since $\dfrac{1}{a+bi} = \dfrac{a-bi}{a^2+b^2}$ and the denominator is positive, this means each of the terms $\dfrac{1}{z_0-r_i}$ has a positive imaginary part, so their sum does as well.
   - We conclude that $\dfrac{p'(z_0)}{p(z_0)}$ has a positive imaginary part: thus, it is not zero, hence the numerator $p'(z_0)$ is not zero.

   (c) Show that if all zeroes of $p(z)$ lie on one side of a line in $\mathbb{C}$, then all zeroes of $p'(z)$ also lie on the same side of that line. Deduce the Gauss-Lucas theorem. [Hint: Use (b) and observe that the result still holds if you translate or rotate the complex plane.]

   - If we translate z_0 and the roots r_i by a fixed constant $\alpha = it$, then the expression $\dfrac{p'(z)}{p(z)} = \dfrac{1}{z-r_1} + \dfrac{1}{z-r_2} + \cdots + \dfrac{1}{z-r_n}$ is left unchanged, but the real axis is shifted vertically to the line $\text{Im}(z) = it$.
   - Applying (b), we see that if the imaginary parts of $r_1, \dots, r_n$ lie above the line while z_0 lies below the line, then $p'(z_0) \neq 0$.
   - We conclude that the result of (b) holds for any horizontal line $\text{Im}(z) = it$: if the roots lie above the line while z_0 lies below, then $p'(z_0) \neq 0$.
   - Now, starting with any horizontal line, if we scale z_0 and the roots r_i by a rotation constant $\alpha = e^{i\theta}$ then the expression $\dfrac{p'(z)}{p(z)}$ is scaled by $e^{-i\theta}$. Thus, if the roots lie on one side of the line and z_0 lies on the other side, then $p'(z_0)$ is still nonzero, since scaling by $e^{-i\theta}$ does not affect being equal to zero.
   - Since we may obtain any line from the real axis by a translation and rotation, we conclude that if all zeroes of $p(z)$ lie on one side of a line in $\mathbb{C}$, then any zero of $p'(z)$ must also lie on the same side of that line.
   - Finally, applying this fact to each of the lines formed by the sides of the smallest convex polygon containing all of the roots $r_1, \dots, r_n$, we see that since each of the roots all lie on the same side of each of these lines, so do the roots of $p'(z)$, and thus the roots of $p'(z)$ lie inside the resulting polygon.
