E. Dummit's Math 4555 ∼ Complex Analysis, Fall 2025 ∼ Homework 2, due Fri Sep 19th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each complex function, calculate its partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$, and determine whether the complex derivative f' exists on any open region R.
   (a) $f(z) = z^4 + z$.
   (b) $f(z) = \overline{z^4 + z}$.
   (c) $f(z) = 3z\overline{z}^2 + z^4$.
   (d) $f(z) = \frac{e^z}{\overline{z} - 1}$.

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2. For each complex function, calculate its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and determine whether the complex derivative f' exists using the Cauchy-Riemann equations.
   (a) $f(x+iy) = (2x^2 + y) + (2y^2 - x)i$.
   (b) $f(x+iy) = 4xy + (2y^2 - 2x^2)i$.
   (c) $f(x+iy) = (3 + e^y \sin x) - (e^y \cos x)i$.
   (d) $f(x+iy) = \sin x \cos y - i \cos x \sin y$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is twice differentiable. We define the Laplacian of f to be $\Delta f = \nabla^2 \cdot f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f_{xx} + f_{yy}$, and we say f is harmonic if $\Delta f = 0$ on the entire domain of f.
   (a) Find the Laplacians of $3x - y$, $x^2 - y^2$, e^{x+y}, $e^x \cos y$, $e^y \cos x$, $\frac{1}{x^2 + y^2}$, $\ln(x^2 + y^2)$, and $\tan^{-1}(y/x)$. Which of these are harmonic?
   (b) Suppose $h(z) = f(x,y) + ig(x,y)$ is a function of $z = x + iy$ where f and g are both twice continuously differentiable. Show that $4\frac{\partial^2 f}{\partial z \partial \overline{z}} = \Delta f$. [Hint: Partial derivatives can be interchanged for twice continuously differentiable functions.]
   (c) Suppose $h(z) = f(x,y) + ig(x,y)$ is a holomorphic function of $z = x + iy$ on the region R. Show that f and g are harmonic on R.

   Remark: Part (c) shows that the real and imaginary parts of a holomorphic function are harmonic. As we will show later in the semester, the converse is also broadly true: a harmonic function defined on a sufficiently nice region is necessarily the real (or imaginary) part of a holomorphic function.

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4. Suppose that we define two differential operators $L = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ and $M = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}$ for some constants $a, b, c, d \in \mathbb{C}$, meaning that $Lf = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$ and similarly $Mf = c\frac{\partial f}{\partial x} + d\frac{\partial f}{\partial y}$ for a function f. Show that if $Lz = 1$, $L\overline{z} = 0$, $Mz = 0$, and $M\overline{z} = 1$ for all $z, \overline{z}$, then in fact we must have $a = \frac{1}{2}$, $b = -\frac{i}{2}$, $c = \frac{1}{2}$, $d = \frac{i}{2}$ so that $L = \frac{\partial}{\partial z}$ and $M = \frac{\partial}{\partial \overline{z}}$.

   Remark: The point of this calculation is that our definitions of $L = \frac{\partial}{\partial z}$ and $M = \frac{\partial}{\partial \overline{z}}$ are forced to be the ones we selected if we want them to act on z and $\overline{z}$ in the expected way.

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5. The goal of this problem is to express the "polar" differential operators $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in terms of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$.

   (a) Suppose $f(z)$ is differentiable where $z = re^{i\theta}$. Show that $\frac{\partial f}{\partial r} = \frac{1}{r}\left[z\frac{\partial f}{\partial z} + \overline{z}\frac{\partial f}{\partial \overline{z}}\right]$ and $\frac{\partial f}{\partial \theta} = i\left[z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}}\right]$. [Hint: Note that $\overline{z} = re^{-i\theta}$ and then use the chain rule.]

   (b) Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ for $f(z) = z^2$ and for $f(z) = z^3\overline{z}^3$. Do these agree with the expected expressions for $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ when f is written in terms of r and θ?

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6. Recall that for a positive integer n, the <u>nth roots of unity</u> are the solutions to the equation $z^n = 1$, and are given explicitly by $\{1, e^{2\pi i/n}, e^{4\pi i/n}, \dots, e^{2(n-1)\pi i/n}\} = \{1, \zeta_n, \zeta_n^2, \dots, \zeta_n^{n-1}\}$ where $\zeta_n = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ for shorthand. The goal of this problem is to explore some results about these numbers.

   (a) Give, in explicit $a + bi$ form, the 3rd, 6th, and 8th roots of unity.

   (b) Show that $1/\zeta_n = \zeta_n^{n-1} = \overline{\zeta_n}$.

   (c) Show that $\zeta_n^{n-1} + \zeta_n^{n-2} + \cdots + \zeta_n = -1$. [Hint: What are the roots of $p(z) = (z^n - 1)/(z - 1)$?]

   (d) Let $\alpha = \zeta_5 + \zeta_5^4$ and $\beta = \zeta_5^2 + \zeta_5^3$. Show that $\alpha + \beta = -1$ and $\alpha\beta = -1$ and deduce that α and β are the roots of the quadratic $p(z) = z^2 + z - 1$. Use this along with $\alpha > 0$ to find α and β explicitly.

   (e) Show that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ and use this to give an explicit formula for the 5th root of unity ζ_5.

   (f) Prove that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2}$ and that $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{1}{2}\sqrt{7}$. [Hint: Let $\alpha = \zeta_7 + \zeta_7^2 + \zeta_7^4$ and $\beta = \zeta_7^3 + \zeta_7^5 + \zeta_7^6$ and use a method similar to (d).]

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7. Suppose z and w are complex numbers. The goal of this problem is to study some properties of the rational function $f_w(z) = \frac{z - w}{1 - z\overline{w}}$, which when $|w| < 1$ is called a <u>Möbius transformation</u>.

   (a) Show that $f_w(z) = \frac{z - w}{1 - z\overline{w}}$ is an invertible function with inverse function $f_{-w}(z) = \frac{z + w}{1 + z\overline{w}}$.

   (b) If $z\overline{w} \neq 1$, show that $1 - \left|\frac{z - w}{1 - z\overline{w}}\right|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\overline{w}|^2}$.

   (c) Suppose that $|w| < 1$. Show that if $|z| = 1$ then $|f_w(z)| = |f_{-w}(z)| = 1$ and deduce that f_w is a bijection from the unit circle $|z| = 1$ to itself.

   (d) Suppose that $|w| < 1$. Show that if $|z| < 1$ then $|f_w(z)| < 1$ and $|f_{-w}(z)| < 1$ and deduce that f_w is a bijection on the interior of the unit disc $|z| < 1$.

   Remark: Parts (c) and (d) show that the function f maps the interior of the unit disc to itself, and also maps the disc's boundary to itself. As we will see, this phenomenon of mapping open regions to open regions, and boundaries to boundaries, are both general properties of holomorphic functions.

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8. [Challenge] Suppose that $p(z) = a(z - r_1)(z - r_2)\cdots(z - r_n)$ is a complex polynomial with roots $r_1, \dots r_n \in \mathbb{C}$. The goal of this problem is to prove the <u>Gauss-Lucas theorem</u>: that the roots of the derivative $p'(z)$ all lie inside of the smallest convex polygon that contains all of the roots $r_1, \dots, r_n$ of $p(z)$.

   (a) Show that $\frac{p'(z)}{p(z)} = \frac{1}{z - r_1} + \frac{1}{z - r_2} + \cdots + \frac{1}{z - r_n}$.

   (b) Suppose that the imaginary parts of $r_1, \dots, r_n$ are positive while the imaginary part of z_0 is negative. Show that $p'(z_0) \neq 0$.

   (c) Show that if all zeroes of $p(z)$ lie on one side of a line in $\mathbb{C}$, then all zeroes of $p'(z)$ also lie on the same side of that line. Deduce the Gauss-Lucas theorem. [Hint: Use (b) and observe that the result still holds if you translate or rotate the complex plane.]
