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1. Express the following complex numbers in rectangular $a + bi$ form:

   (a) $(3 + i) - (4 - 2i)(1 - i)$.

   - We have $(3 + i) - (4 - 2i)(1 - i) = (3 + i) - (2 - 6i) = \boxed{1 + 7i}$.

   (b) $(4 + 3i)/(5 - i)$.

   - We have $(4 + 3i)/(5 - i) = \frac{(4+3i)(5+i)}{(5-i)(5+i)} = \boxed{\frac{17}{26} + \frac{19}{26}i}$.

   (c) $4e^{7i\pi/6}$.

   - We have $4e^{7i\pi/6} = 4\cos(7\pi/6) + 4i\sin(7\pi/6) = \boxed{-2\sqrt{3} - 2i}$.

   (d) $e^{i\pi/4} + 3e^{3i\pi/4}$.

   - We have $e^{i\pi/4} + 3e^{3i\pi/4} = [\cos(\pi/4) + 3\cos(3\pi/4)] + i[\sin(\pi/4) + 3\sin(3\pi/4)] = \boxed{-\sqrt{2} + 2\sqrt{2}i}$.

   (e) $e^{2025i\pi/3}$.

   - We have $e^{2025i\pi/3} = \cos(2025\pi/3) + i\sin(2025\pi/3) = \boxed{-1 + 0i}$.

   (f) $(1 + i)^{2024}$.

   - We have $(1 + i)^{2024} = (\sqrt{2}e^{i\pi/4})^{2024} = 2^{1012}e^{506i\pi} = \boxed{2^{1012} + 0i}$.

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2. Express the following complex numbers in exponential $re^{i\theta}$ form:

   (a) $3i$.

   - We have $3i = \boxed{3e^{i\pi/2}}$.

   (b) $-2 - 2i\sqrt{3}$.

   - We have $-2 - 2i\sqrt{3} = \boxed{4e^{4i\pi/3}}$.

   (c) $-1 + i$.

   - We have $-1 + i = \boxed{\sqrt{2}e^{3i\pi/4}}$.

   (d) $\pi + ei$.

   - We have $\pi + ei = \boxed{\sqrt{\pi^2 + e^2}e^{i\cdot\arctan(e/\pi)}}$.

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3. Find all $z \in \mathbb{C}$ satisfying the following equations:

   (a) $z^2 + 2z + 2 = 0$.

   - The quadratic formula yields $z = \frac{-2\pm\sqrt{2^2-4\cdot 2}}{2} = \boxed{-1 \pm i}$.

   (b) $z^2 = 3 + 4i$.

   - Using the square root formula from the notes yields $z = \boxed{\pm(2 + i)}$.

   (c) $z^2 - (2 - i)z - 2i = 0$.

   - The quadratic formula yields $z = \frac{(2-i)\pm(3+4i)}{2} = \frac{(2-i)\pm(2+i)}{2} = \boxed{2, -i}$.

   (d) $z^3 = -1$.

   - The root formula yields $z = \boxed{e^{i\pi/3}, e^{3i\pi/3}, e^{5i\pi/3}} = \{\frac{1+i\sqrt{3}}{2}, -1, \frac{1-i\sqrt{3}}{2}\}$.

(e) $z^8 = 1$.

- The root formula yields $z = \boxed{e^{2ki\pi/8} \text{ for } 0 \leq k \leq 7} = \{1, \frac{1+i}{\sqrt{2}}, i, \frac{-1+i}{\sqrt{2}}, -1, \frac{-1-i}{\sqrt{2}}, -i, \frac{1-i}{\sqrt{2}}\}$.

(f) $z^4 = 3$.

- The root formula yields $z = \boxed{3^{1/4} e^{2ki\pi/4} \text{ for } 0 \leq k \leq 3} = \{3^{1/4}, 3^{1/4}i, -3^{1/4}, -3^{1/4}i\}$.

(g) $e^z = 1$.

- By periodicity we have $z = \boxed{2k\pi i \text{ for integers } k}$.

(h) $e^z = 1 - i\sqrt{3}$.

- Converting to polar yields $e^z = 2e^{-i\pi/3}$ so $z = \boxed{\ln(2) - \frac{i\pi}{3} + 2k\pi i \text{ for integers } k}$.

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4. Plot each of the given regions in the complex plane (you may want to use a computer). For each region, identify whether it is (i) open, (ii) closed, (iii) connected, and (iv) bounded.

(a) $|z-1| < 1$.

- $|z-1| < 1$ is open, not closed, connected, bounded.

(b) $1 \leq |z| \leq 3$.

- $1 \leq |z+1| \leq 2$ is not open, closed, connected, bounded.

(c) $|z| > |z-1|$.

- $|z| > |z-1|$ is open, not closed, connected, not bounded.

(d) $0 < \text{Im}(z) \leq 1$.

- $0 < \text{Im}(z) \leq 1$ is not open, not closed, connected, not bounded.

(e) $\text{Re(z)} \leq \text{Im(z)}$.

- $\text{Re(z)} \leq \text{Im(z)}$ is not open, closed, connected, not bounded.

(f) $\text{Re}(z) \cdot \text{Im}(z) > 1$.

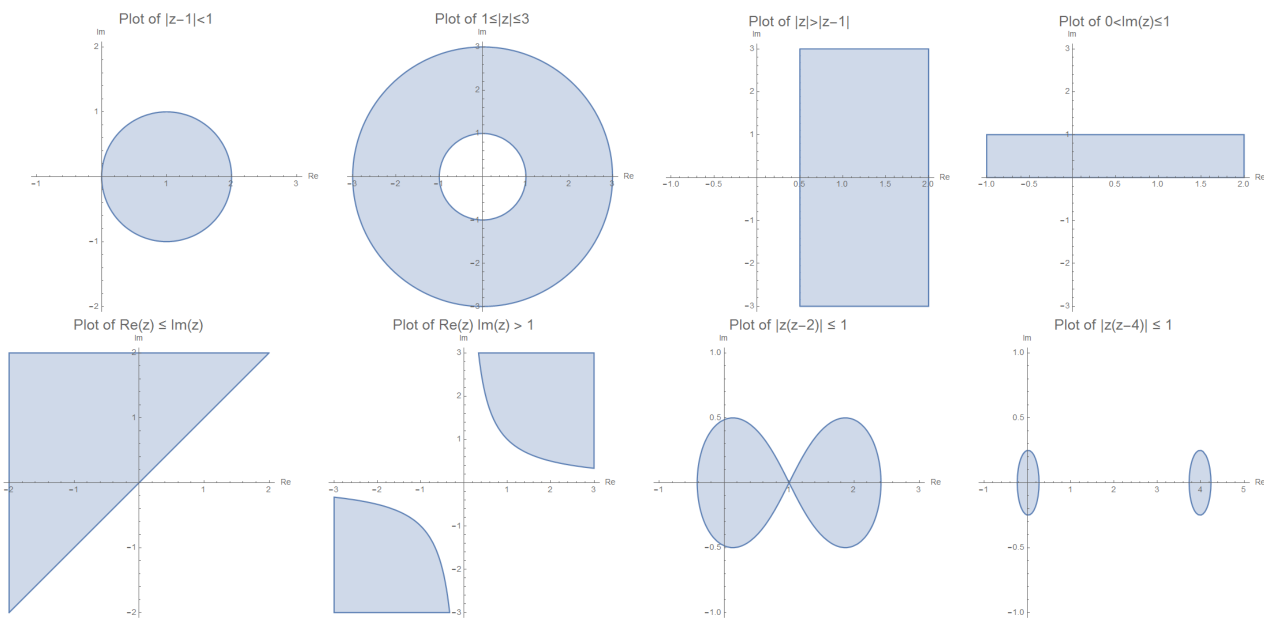
- $\text{Re}(z) \cdot \text{Im}(z) > 1$ is open, not closed, not connected, not bounded.

(g) $|z(z-2)| \leq 1$.

- $|z(z-2)| \leq 1$ is not open, closed, connected, bounded.

(h) $|z(z-4)| \leq 1$.

- $|z(z-4)| \leq 1$ is not open, closed, not connected, bounded.



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5. Compute the following complex limits or show they do not exist:

(a) $\lim_{z\to 1} \dfrac{z^3-1}{z-1}$.

- We have $\lim_{z\to 1} \dfrac{z^3-1}{z-1} = \lim_{z\to 1} \dfrac{(z-1)(z^2+z+1)}{z-1} = \lim_{z\to 1}(z^2+z+1) = \boxed{3}$.

(b) $\lim_{z\to i} \dfrac{z^3-1}{z-1}$.

- We have $\lim_{z\to 1} \dfrac{z^3-1}{z-1} = \dfrac{i^3-1}{i-1} = \boxed{i}$.

(c) $\lim_{z\to 0} \dfrac{z}{|z|}$.

- This limit $\boxed{\text{does not exist}}$. In fact, it does not even exist along the real axis, since $\lim_{t\to 0} \dfrac{t}{|t|}$ is -1 along the negative real axis and $+1$ along the positive real axis.

(d) $\lim_{z\to 0} \dfrac{z^3}{|z|^2}$.

- Note that $\left|\dfrac{z^3}{|z|^2}\right| = \dfrac{|z|^3}{|z|^2} = |z|$, so $\lim_{z\to 0} |f(z)| = \lim_{z\to 0} |z| = 0$.
- It is not hard to see using the definition of limit that if $\lim_{z\to 0} |f(z)| = 0$ then $\lim_{z\to 0} f(z)$ exists and equals zero as well. So the limit exists and is $\boxed{0}$.

(e) $\lim_{z\to 1} \dfrac{z^2-1}{z^2+z-2\overline{z}}$.

- This limit $\boxed{\text{does not exist}}$.
- Along a horizontal line with $z = 1+t$ as $t \to 0$ we have $\lim_{t\to 0} \dfrac{(1+t)^2-1}{(1+t)^2+(1+t)-2(1+t)} = \lim_{t\to 0} \dfrac{t(t+2)}{t(t+1)} = \lim_{t\to 0} \dfrac{t+2}{t+1} = 2$, while along a vertical line with $z = 1+it$ as $t \to 0$ we have $\lim_{t\to 0} \dfrac{(1+it)^2-1}{(1+it)^2+(1+it)-2(1-it)} = \lim_{t\to 0} \dfrac{t(-t+2i)}{t(-t+5i)} = \lim_{t\to 0} \dfrac{-t+2i}{-t+5i} = \dfrac{2}{5}$.

(f) $\lim_{z\to 0} \dfrac{z^{720}}{|z|^{720}}$.

- This limit $\boxed{\text{does not exist}}$. Along the line $z = e^{i\theta}t$ for an arbitrary angle θ we obtain $\lim_{z\to 0} \dfrac{e^{720i\theta}t^{720}}{t^{720}} = e^{720i\theta}$ which for different values of θ (e.g., $\theta = 0$ and $\theta = \pi/720$) can take different values.

(g) $\lim_{z\to 0} \dfrac{\mathrm{Re}(z)\cdot \mathrm{Im}(z)^2}{\mathrm{Re}(z)^2+\mathrm{Im}(z)^4}$. [Hint: Try the path $z = t^2+it$ as $t \to 0$.]

- This limit $\boxed{\text{does not exist}}$. Along the real axis $z = t+0i$ as $t \to 0$ the limit is $\lim_{t\to 0} \dfrac{0}{t^2} = 0$ while along the path $z = t^2+it$ as $t \to 0$ the limit is $\lim_{t\to 0} \dfrac{t^4}{2t^4} = \dfrac{1}{2}$.

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6. The goal of this problem is to illustrate some uses of complex exponentials for trigonometry. You may assume in this problem that indefinite integrals involving complex parameters behave the same way as if the parameters were real (we will later prove this), and you may use Euler's identity $e^{ix} = \cos x + i\sin x$.

(a) If x is real, show that $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$.

- From Euler's identity we have $e^{ix} = \cos x + i\sin x$ and so $e^{-ix} = \cos x - i\sin x$.

- Adding and subtracting the equations gives $e^{ix} + e^{-ix} = 2\cos x$ and $e^{ix} - e^{-ix} = 2i\sin x$, and then rescaling yields the desired formulas.

(b) Compute $\int \cos^4 x\,dx$ and $\displaystyle\int \sin^4 x\,dx$. [Hint: Use (a).]

- We have $\cos^4 x = \left[\dfrac{e^{ix}+e^{-ix}}{2}\right]^4 = \dfrac{1}{16}\left[e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix}\right] = \dfrac{1}{8}[\cos 4x + 4\cos 2x + 6]$.

  Integrating then yields $\int \cos^4 x\,dx = \int \frac{1}{8}(\cos 4x + 4\cos 2x + 6)\,dx = \boxed{\frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{4}x + C}$.

- Likewise, $\sin^4 x = \left[\dfrac{e^{ix}-e^{-ix}}{2i}\right]^4 = \dfrac{1}{16}\left[e^{4ix} - 4e^{2ix} + 6 - 4e^{-2ix} + e^{-4ix}\right] = \dfrac{1}{8}[\cos 4x - 4\cos 2x + 6]$.

  Integrating then yields $\int \sin^4 x\,dx = \int \frac{1}{8}(\cos 4x - 4\cos 2x + 6)\,dx = \boxed{\frac{1}{32}\sin 4x - \frac{1}{4}\sin 2x + \frac{3}{4}x + C}$.

(c) Compute $\displaystyle\int e^{ax}\cos bx\,dx$ and $\displaystyle\int e^{ax}\sin bx\,dx$. [Hint: Take real and imaginary parts of $\displaystyle\int e^{(a+bi)x}\,dx$.]

- Note $\int e^{(a+bi)x}\,dx = \dfrac{e^{(a+bi)x}}{a+bi} + C = \dfrac{(a-bi)e^{ax}(\cos bx + i\sin bx)}{a^2+b^2} + C = \dfrac{a\cos bx + b\sin bx}{a^2+b^2}e^{ax} + \dfrac{-b\cos bx + a\sin bx}{a^2+b^2}e^{ax}i + C$.
- The real part is $\mathrm{Re}[\int e^{(a+bi)x}\,dx] = \int \mathrm{Re}[e^{(a+bi)x}]\,dx = \int e^{ax}\cos bx\,dx$ and similarly the imaginary part is $\int e^{ax}\sin bx\,dx$.
- Thus, $\displaystyle\int e^{ax}\cos bx\,dx = \boxed{\frac{a\cos bx + b\sin bx}{a^2+b^2}e^{ax} + C}$ while $\displaystyle\int e^{ax}\sin bx\,dx = \boxed{\frac{-b\cos bx + a\sin bx}{a^2+b^2}e^{ax} + C}$.

(d) Prove that $1 + e^{ix} + e^{2ix} + \cdots + e^{inx} = \dfrac{e^{(n+1)ix}-1}{e^{ix}-1} = e^{(n/2)ix}\dfrac{\sin[\frac{n+1}{2}x]}{\sin(x/2)}$ for any positive integer n and any $0 < x < 2\pi$.

- First, note that $1 + e^{ix} + e^{2ix} + \cdots + e^{inx}$ is a geometric series with $r = e^{ix}$, and that $r \neq 1$ since $0 < x < 2\pi$. So by the usual formula, the sum is $1 + r + r^2 + \cdots + r^n = \dfrac{r^{n+1}-1}{r-1} = \dfrac{e^{(n+1)ix}-1}{e^{ix}-1}$. (For completeness, the formula follows immediately from the difference-of-powers identity $1 - r^{n+1} = (1-r)(1+r+r^2+\cdots+r^n)$.)
- For the second formula, observe that $\dfrac{e^{(n+1)ix}-1}{e^{ix}-1} = \dfrac{e^{[(n+1)/2]ix}}{e^{ix/2}}\dfrac{[e^{[(n+1)/2]ix} - e^{-[(n+1)/2]ix}]/(2i)}{[e^{ix/2} - e^{-ix/2}]/(2i)} = e^{(n/2)ix}\dfrac{\sin[\frac{n+1}{2}x]}{\sin(x/2)}$ using (a).

(e) Deduce that $1 + \cos x + \cos 2x + \cdots + \cos nx = \dfrac{\sin[\frac{n+1}{2}x]\cos[\frac{n}{2}x]}{\sin(x/2)}$ and $\sin x + \sin 2x + \cdots + \sin nx = \dfrac{\sin[\frac{n+1}{2}x]\sin[\frac{n}{2}x]}{\sin(x/2)}$ for any positive integer n and any $0 < x < 2\pi$.

- Simply take the real and imaginary parts of the identity from (d): the real part of the LHS is $1 + \cos x + \cos 2x + \cdots + \cos nx$ while the real part of the RHS is $\dfrac{\sin[\frac{n+1}{2}x]\cos[\frac{n}{2}x]}{\sin(x/2)}$, and the imaginary part of the LHS is $\sin x + \sin 2x + \cdots + \sin nx$ while the imaginary part of the RHS is $\dfrac{\sin[\frac{n+1}{2}x]\sin[\frac{n}{2}x]}{\sin(x/2)}$.

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7. The real numbers are an example of an ordered field, which is a field containing a subset P (the "positive" elements) such that (i) P is closed under addition, (ii) P is closed under multiplication, and (iii) every nonzero element of the field is either in P or its additive inverse is in P but not both. Prove that $\mathbb{C}$ is not an ordered field for any possible choice of P. [Hint: Consider i.]

- Suppose $\mathbb{C}$ is an ordered field. By (iii), since $i \neq 0$, either $i \in P$ or $-i \in P$.

- If $i \in P$ then since $i^3 = -i$ by (ii) we would have $-i \in P$, which contradicts (iii).
- But if $-i \in P$ then since $(-i)^3 = i$ by (ii) we would have $i \in P$, which also contradicts (iii).
- In either case we have a contradiction, so $\mathbb{C}$ cannot be an ordered field.

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8. The goal of this problem is to illustrate one of the original historical applications of the complex numbers: that of solving the cubic equation.

   (a) Suppose that $z^3 + az^2 + bz + c = 0$. Show that $t = z + a/3$ has $t^3 + pt + q = 0$ where $p = b - a^2/3$ and $q = (2/27)a^3 - ab/3 + c$. Thus, it suffices to solve cubics of the form $t^3 + pt + q = 0$.

   - We have

$$\begin{aligned} t^3 + pt + q &= (z + a/3)^2 + (b - a^2/3)(z + a/3) + [(2/27)a^3 - ab/3 + c] \\ &= [z^3 + az^2 + (a^2/3)z + (a^3/27)] + [bz - (a^2/3)z + ab/c - a^3/9] + [(2/27)a^3 - ab/3 + c] \\ &= z^3 + az^2 + bz + c = 0 \end{aligned}$$

   as claimed.

   (b) Suppose that $t^3 + pt + q = 0$. Define new variables x and y such that $x + y = t$ and $3xy = -p$. Show that $x^3 + y^3 = -q$ and then solve for x^3 and y^3.

   - First, we have $x^3 + y^3 = (x + y)^3 - 3xy(x + y) = t^3 + pt = -q$.
   - The equation $3xy = -p$ implies $y = -p/(3x)$, and then $x^3 + y^3 = -q$ becomes $x^3 - p^3/(27x^3) = -q$, whence $x^6 + qx^3 - \dfrac{p^3}{27} = 0$.
   - This is a quadratic in x^3, so solving yields $x^3 = -\dfrac{q}{2} \pm \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}$ and then $y^3 = -q - x^3 = -\dfrac{q}{2} \mp \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}$.

   (c) Conclude that the solutions to the cubic $t^3 + pt + q = 0$ are the three numbers of the form $t = A + B$, with $A = \sqrt[3]{-\dfrac{q}{2} + \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}}$ and $B = \sqrt[3]{-\dfrac{q}{2} - \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}}$, where the cube roots are selected so that $AB = -p/3$. These formulas are known as <u>Cardano's formulas</u>.

   - Combining (b) and (c) shows that $t = x + y$ where $3xy = -p$, $x^3 = -\dfrac{q}{2} \pm \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}$, and $y^3 = -\dfrac{q}{2} \mp \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}$.
   - Letting A^3 be the term with the plus sign, we obtain $t = A + B$ where $AB = -p/3$ and $A^3 = -\dfrac{q}{2} + \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}$ and $B^3 = -\dfrac{q}{2} - \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}$.
   - Taking cube roots yields the desired $t = A+B$, with $A = \sqrt[3]{-\dfrac{q}{2} + \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}}$ and $B = \sqrt[3]{-\dfrac{q}{2} - \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}}$ where $AB = -p/3$. Since there are three possible choices of cube root for A (and then B is determined uniquely) these yield the three possible roots of the cubic.

   (d) Verify that the cubic $f(t) = t^3 - 15t - 4$ has three real roots and that they are 4 and $-2 \pm \sqrt{3}$.

   - We have $f(4) = 4^3 - 15 \cdot 4 - 4 = 0$ so 4 is a root. Factoring yields $t^3 - 15t - 4 = (t - 4)(t^2 + 4t + 1)$ and the quadratic has roots $-2 \pm \sqrt{3}$ by the quadratic formula.
   - Since a cubic cannot have more than three roots, $f(t)$ has only the three given roots, which are clearly real.

   (e) Use Cardano's formulas to find the roots of $f(t) = t^3 - 15t - 4$, and then show that they do simplify to yield the same answers in (d) using the calculation $(2 + i)^3 = 2 + 11i$.

   - Plugging into Cardano's formulas yields the roots $t = A + B$ where $A^3 = 2 + \sqrt{-121}$ and $B^3 = 2 - \sqrt{-121}$ and $AB = 5$.

- Noting that $(2+i)^3 = 2+11i$, we see that one possible cube root of $2+\sqrt{-121}$ is $A = 2+i$, which yields a corresponding value $B = \dfrac{5}{2+i} = 2-i$, in which case $t = (2+i)+(2-i) = 4$.
- Another possible cube root is $A = (2+i)e^{2\pi i/3} = \frac{-\sqrt{3}-2}{2} + \frac{2\sqrt{3}-1}{2}i$ which has a corresponding $B = (2-i)e^{-2\pi i/3} = \frac{-\sqrt{3}-2}{2} - \frac{2\sqrt{3}-1}{2}i$, in which case $t = (\frac{-\sqrt{3}-2}{2} + \frac{\sqrt{3}-2}{2}i) + \frac{-\sqrt{3}-2}{2} - \frac{\sqrt{3}-2}{2}i = -2-\sqrt{3}$.
- The third possible cube root is $A = (2+i)e^{4\pi i/3} = \frac{\sqrt{3}-2}{2} + \frac{-2\sqrt{3}-1}{2}i$ which has a corresponding $B = (2-i)e^{-2\pi i/3} = \frac{\sqrt{3}-2}{2} - \frac{-2\sqrt{3}-1}{2}i$, in which case $t = (\frac{\sqrt{3}-2}{2} + \frac{-2\sqrt{3}-1}{2}i) + \frac{\sqrt{3}-2}{2} - \frac{-2\sqrt{3}-1}{2}i = -2+\sqrt{3}$.
- We do obtain the same three roots as in (d), as claimed.

- Remark: The calculation in (e) was performed by Bombelli in 1572. This rather perplexing appearance of square roots of negative numbers in the formulas for real solutions to cubic equations was the original impetus that led to the development and acceptance of complex numbers in mathematics (although unsurprisingly, it did take a while!).

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9. [Challenge] The goal of this problem is to justify the remark following problem 8 by showing that for *any* cubic polynomial whose roots are real, Cardano's formulas necessarily involve non-real radicals. Suppose $f(z) = z^3 + az^2 + bz + c = (z-r_1)(z-r_2)(z-r_3)$ is a cubic polynomial with real coefficients $a, b, c \in \mathbb{R}$ and complex roots $r_1, r_2, r_3 \in \mathbb{C}$. Define its discriminant as $\Delta = (r_1-r_2)^2(r_1-r_3)^2(r_2-r_3)^2$ and let $p = b - a^2/3$ and $q = (2/27)a^3 - ab/3 + c$.

(a) Show that $\Delta = 0$ when f has a repeated root, that $\Delta > 0$ when f has three distinct real roots, and that $\Delta < 0$ when f has a pair of nonreal complex-conjugate roots.

- Clearly $\Delta = 0$ occurs if and only if one of the terms $r_1 - r_2$, $r_1 - r_3$, $r_2 - r_3$ is zero, which happens precisely when two of the roots of f are equal.
- If f has three distinct real roots, then each of the squares $(r_i - r_j)^2$ is positive so their product Δ is also positive.
- Finally, suppose f has a real root r and two complex-conjugate roots $x \pm yi$. Then $\Delta = (r-x-yi)^2(r-x+yi)^2(2yi)^2 = -4y^2[(r-x)^2+y^2]$ which is negative because $y \neq 0$.

(b) Show that the discriminant of $f(t)$ is the same as that of $g(t) = t^3 + pt + q$ where $p = b - a^2/3$ and $q = (2/27)a^3 - ab/3 + c$.

- Note that adding a constant to all three roots does not change Δ, since it is a product of differences of the roots. Thus, using the translation in 8(b) does not change the discriminant, but shifts $f(t)$ to $g(t)$, so their discriminants are equal.

(c) Show that the discriminant of $f(t)$ equals $-27p^3 - 4q^2$. [Hint: $\Delta$is a symmetric function in r_1, r_2, r_3 and must therefore be a polynomial in the three functions $r_1 + r_2 + r_3 = 0$, $r_1r_2 + r_1r_3 + r_2r_3 = p$, and $r_1r_2r_3 = -q$. Since it has degree 6, it must be of the form $Xp^3 + Yq^2$. Plug in values for r_1, r_2, r_3 subject to $r_1 + r_2 + r_3 = 0$ to find the coefficients.]

- Following the hint, because $\Delta$is a symmetric function in r_1, r_2, r_3 of degree 6, it is a polynomial of total degree 6 in the three elementary symmetric functions $\sigma_1 = r_1+r_2+r_3$, $\sigma_2 = r_1r_2+r_1r_3+r_2r_3$, and $\sigma_3 = r_1r_2r_3$.
- By using the translation in (b), we may shift to make $\sigma_1 = 0$. Then since σ_2 has degree 2 and σ_3 has degree 3, the only terms of degree 6 are σ_2^3 and σ_3^2. Since $\sigma_2 = p$ and $\sigma_3 = -q$ by Vieta's formulas, that means we must have $\Delta = Xp^3 + Yq^2$ for some coefficients X and Y.
- Setting $r_1 = 0$, $r_2 = 1$, $r_3 = -1$ yields $p = -1$, $q = 0$, $\Delta = 4$, so $4 = -X$.
- Setting $r_1 = r_2 = 1$ and $r_3 = -2$ yields $p = -3$, $q = 2$, $\Delta = 0$, so $0 = -27X + 4Y$ hence $Y = -27$.
- We deduce that $\Delta = -27p^3 - 4q^2$ as claimed.

(d) Show that if f has three distinct real roots, then Cardano's formulas require computing cube roots of non-real complex numbers.

- By part (c) above, the term under the square root in the $-\dfrac{q}{2} + \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}$ expression in Cardano's formulas from 8(c) is $-\Delta/108$. By (a), if the polynomial f has three real roots, then the discriminant Δ is positive, so the square root $\sqrt{-\Delta/108}$ is nonreal, as claimed.
