E. Dummit's Math 4555 ∼ Complex Analysis, Fall 2025 ∼ Homework 9, due Fri Dec 5th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Describe the image of each region under the given mapping:
   (a) The closed disc $|z-2| \leq 2$ under $f(z) = 1/z$.
   (b) The closed disc $|z-2| \leq 2$ under $f(z) = 4/(z-2)$.
   (c) The upper half-plane $\text{Re}(z) > 0$ under $f(z) = \frac{z+i}{z-i}$.
   (d) The upper half-plane $\text{Re}(z) > 0$ under $f(z) = z^2$. [Hint: Use polar.]
   (e) The open disc $|z| < 1$ under $f(z) = \frac{z+5}{z+2}$.
   (f) The first quadrant $\text{Re}(z) \geq 0$, $\text{Im}(z) \geq 0$ under $f(z) = 2/(z+i)$.

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2. For each set of properties, find a fractional linear transformation $T(z) = \dfrac{az+b}{cz+d}$ satisfying them:
   (a) $T(1) = 0$, $T(2) = 1$, $T(3) = \infty$.
   (b) $T(i) = 1$, $T(-i) = 3$, $T(\infty) = \infty$.
   (c) $T(0) = 1$, $T(1) = \infty$, $T(\infty) = 0$.
   (d) $T^3 = T \circ T \circ T$ is the identity, but T is not linear. [Hint: One of the functions above works.]
   (e) T maps the unit circle to itself and also has $T(1/2) = 0$. [Hint: It is an automorphism of the disc.]

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3. Let R be the right half-plane $\text{Re}(z) > 0$ and let D be the unit disc $|z| < 1$.
   (a) Construct an analytic isomorphism from R to D.
   (b) Let S be the half-strip $-\dfrac{\pi}{2} < \text{Im}(z) < \dfrac{\pi}{2}$, $\text{Re}(z) > 0$. Find the image of the boundary of S, and of S itself, under the analytic isomorphism $f(z) = \sinh z$.
   (c) Construct an analytic isomorphism from S to D.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. For each f on each region, determine (with rigorous justification) the number of zeroes f has in the region:
   (a) $f(z) = z^{12} - 4z^8 + 9z^5 - 2z + 1$ on the region $|z| < 1$.
   (b) $f(z) = z^7 - 7z + 3$ on the region $|z| < 1$.
   (c) $f(z) = z^7 - 7z + 3$ on the region $1 < |z| < 2$.
   (d) $f(z) = 3z^4 + z^3 \sin z - 1$ on the region $|z| < 1$. [Hint: Use $|\sin(x+iy)| \leq \cosh(y)$.]

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5. Suppose that $f(z)$ is an entire function.
   (a) If there exists a line L in the complex plane such that $f(z)$ lies on L for all $z \in \mathbb{C}$, show that f must be constant.
   (b) If $|f|$ is constant, show that f must be constant.
   (c) If $\text{Re}(f)$ is bounded, show that f must be constant. [Hint: Consider e^f.]

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6. The goal of this problem is to prove that if f is a nonzero holomorphic function on a simply connected region R, then there exists a holomorphic function g on R with $f(z) = e^{g(z)}$ for all $z \in R$.

   (a) Suppose f is nonzero and holomorphic on R. Show that $h(z) = \frac{f'(z)}{f(z)}$ is also holomorphic on R, and deduce that it has an antiderivative $H(z)$ on R.

   (b) With $H(z)$ as in part (a), show that there exists a nonzero constant C such that $e^{H(z)} = Cf(z)$ for all $z \in R$. [Hint: Differentiate $f(z)e^{-H(z)}$.]

   (c) Deduce that there exists a holomorphic function g on R with $f(z) = e^{g(z)}$ for all $z \in R$.

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7. The goal of this problem is to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2+1} = \frac{\pi}{2}\coth\pi - \frac{1}{2}$ and $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Let $\alpha \in \mathbb{C}$ with $\alpha \neq k\pi$ for any integer k.

   (a) Show that for $z = x + iy$ we have $|\cot z|^2 = \dfrac{\cot^2 x \coth^2 y + 1}{\cot^2 x + \coth^2 y}$.

   (b) For each positive integer n, let γ_n be the counterclockwise boundary of the square with vertices $(\pm 1 \pm i)\pi(n + \frac{1}{2})$. Show that $\int_{\gamma_n} \dfrac{\cot z}{z(z-\alpha)}\, dz \to 0$ as $n \to \infty$. [Hint: Estimate the integrand on each side.]

   (c) With γ_n as in (b), if $n > |\alpha|$ show $\dfrac{1}{2\pi i}\displaystyle\int_{\gamma_n} \dfrac{\cot z}{z(z-\alpha)}\, dz = \sum_{k=1}^{n} \dfrac{1}{k\pi(k\pi - \alpha)} + \sum_{k=1}^{n} \dfrac{1}{k\pi(k\pi+\alpha)} + \dfrac{\cot\alpha}{\alpha} - \dfrac{1}{\alpha^2}$.

   (d) Deduce that $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k^2\pi^2 - \alpha^2} = -\frac{\cot\alpha}{\alpha} + \frac{1}{\alpha^2}$ for all $\alpha \neq k\pi$ for integers k.

   (e) Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1} = \frac{\pi}{2}\coth\pi - \frac{1}{2}$.

   (f) With γ_n as in (b), show that $\dfrac{1}{2\pi i}\displaystyle\int_{\gamma_n} \frac{\cot z}{z^2}\, dz = \sum_{k=1}^{n} \frac{2}{k^2\pi^2} - \frac{1}{3}$.

   (g) Prove that $\zeta(2) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

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8. [Challenge] The goal of this problem is to prove Euler's product formula for sine: $\dfrac{\sin z}{z} = \displaystyle\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right)$.

   (a) Show that $\cot z - \dfrac{1}{z} = \sum_{k=1}^{\infty} \dfrac{2z}{z^2 - k^2\pi^2}$ for all $z \neq k\pi$.

   (b) Suppose R is a closed bounded region not containing any point $k\pi$ for integers k. Show that the series $\sum_{k=1}^{\infty} \dfrac{2z}{z^2 - k^2\pi^2}$ converges absolutely and uniformly on R.

   (c) Prove that $\operatorname{Log}\dfrac{\sin z}{z} = \sum_{k=1}^{\infty} \operatorname{Log}\left[1 - \dfrac{z^2}{k^2\pi^2}\right] + C$ for some constant C and $z \neq k\pi$.

   (d) Deduce Euler's product formula $\dfrac{\sin z}{z} = \prod_{k=1}^{\infty}(1 - \dfrac{z^2}{k^2\pi^2})$ for all $z \in \mathbb{C}$.

   (e) Prove that $\sum_{k=1}^{\infty} \dfrac{1}{k^2} = \dfrac{\pi^2}{6}$. [Hint: Compare series expansions.]

   Remark: In fact, Euler's original evaluation of $\zeta(2)$ used precisely this argument of comparing the series expansions of $\frac{\sin z}{z}$ and $\prod_{k=1}^{\infty}(1 - \frac{z^2}{k^2\pi^2})$. Intuitively, $\frac{\sin z}{z}$ has simple zeroes at $z = k\pi$ for each integer k, so if $\frac{\sin z}{z}$ were a polynomial, it would have a "factorization" $\frac{\sin z}{z} = \prod_{k=1}^{\infty}(1 - \frac{z^2}{k^2\pi^2})$. Indeed, another approach to proving this product formula is to show that the product converges to an entire function, then observing that the ratio $\frac{\sin z}{z} \Big/ \prod_{k=1}^{\infty}(1 - \frac{z^2}{k^2\pi^2})$ has no zeroes and no poles hence is the exponential of an entire function by problem 6. By estimating growth rates, one can show that the exponential must actually be constant, thus yielding Euler's formula.
