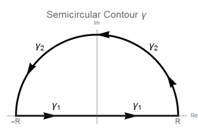
1. Evaluate the following real integrals, making sure to explain all steps (e.g., introducing contours, calculating residues, defining branch cuts, bounding integrands, etc.).

   (a) $\int_0^{2\pi} \dfrac{1}{20\sin\theta + 25}\,d\theta$.

   - Using the method from class we calculate $f(z) = r(\dfrac{z+z^{-1}}{2}, \dfrac{z-z^{-1}}{2i}) \cdot \dfrac{1}{iz} = \dfrac{1}{10z^2+25iz-10}$ which has simple poles at $z = -i/2, -2i$. The only one of these inside the unit circle is $z = -i/2$, and the residue of f there is $\lim_{z\to z_0} \dfrac{1}{20z+25i} = \dfrac{1}{15i}$.
   - Hence by the residue theorem we see that $\int_0^{2\pi} \dfrac{1}{20\sin\theta+25}\,d\theta = \int_\gamma f(z)\,dz = 2\pi i \cdot (\dfrac{1}{15i}) = \boxed{\dfrac{2\pi}{15}}$.

   (b) $\int_{-\infty}^{\infty} \dfrac{x^2}{(x^2+4)^4}\,dx$.

   - Take $f(z) = \dfrac{z^2}{(z^2+4)^4}$ and the upper-semicircle contour:



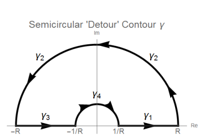
   - Then γ encloses the pole of order 4 at $z = 2i$ with residue $\dfrac{1}{3!}\lim_{z\to 2i}\dfrac{d^3}{dz^3}[(z-2i)^3 f(z)] = \dfrac{1}{3!}[\lim_{z\to 2i}[-24(z+2i)^{-4} + 120z(z+2i)^{-5} - 120z^2(z+2i)^{-6}] = -\dfrac{i}{1024}$, so by the residue theorem we have $\int_\gamma f(z)\,dz = 2\pi i \cdot \text{Res}_f(2i) = \dfrac{\pi}{512}$.
   - On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \dfrac{t^2}{(t^2+4)^4}\,dt \to I$ as $R \to \infty$.
   - On γ_2 we have $|f(z)| = O(R^{-5})$ so since the arclength of γ_2 is πR we see $\int_{\gamma_2} f(z)\,dz = O(R^{-4}) \to 0$.
   - So taking $R \to \infty$ yields $I = \boxed{\dfrac{\pi}{512}}$.

   (c) $\int_{-\infty}^{\infty} \dfrac{\cos 2x}{x^2+1}\,dx$.

   - Take $f(z) = \dfrac{e^{2iz}}{z^2+1}$ and the upper-semicircle contour.
   - Then γ encloses the simple pole at $z = i$ with residue $\lim_{z\to i}(z-i)f(z) = \dfrac{e^{-2}}{2i}$, so by the residue theorem we have $\int_\gamma f(z)\,dz = 2\pi i \cdot \text{Res}_f(i) = \pi/e^2$.
   - On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \dfrac{e^{2iz}}{z^2+1}\,dt \to \int_{-\infty}^{\infty} \dfrac{\cos 2x}{x^2+1}\,dx + i\int_{-\infty}^{\infty} \dfrac{\sin 2x}{x^2+1}\,dx$ as $R \to \infty$.
   - On γ_2 we have $\left|f(Re^{it})\right| \le \dfrac{e^{-2R\sin t}}{R^2-1} = O(R^{-2})$ so since the arclength of γ_2 is πR we see $\int_{\gamma_2} f(z)\,dz = O(R^{-1}) \to 0$.
   - So taking $R \to \infty$and extracting the real part yields $I = \boxed{\pi/e^2}$.

   (d) $\int_{-\infty}^{\infty} \dfrac{\sin 3x}{x(x^2+1)}\,dx$.

   - Take $f(z) = \dfrac{e^{3iz}}{z(z^2+1)}$ and a contour that detours around the pole at $z = 0$:



- Then γ encloses the simple pole at $z = i$ with residue $\lim_{z\to i}(z-i)f(z) = -\dfrac{1}{2}e^{-3}$, so by the residue theorem we have $\int_\gamma f(z)\,dz = 2\pi i \cdot \operatorname{Res}_f(i) = -\pi i e^{-3}$.
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{1/R}^{R} \dfrac{e^{3it}}{t(t^2+1)}\,dt \to \int_0^\infty \dfrac{e^{2it}}{t(t^2+1)}\,dt$ as $R \to \infty$.
- On γ_2 we have $|f(z)| = \left|\dfrac{e^{3iR(\cos t + i \sin t)}}{(Re^{it})(R^2e^{2it}+1)}\right| \le \dfrac{e^{-2R\sin t}}{R(R^2-1)} = O(R^{-3})$ so since the arclength of γ_2 is πR we see $\int_{\gamma_2} f(z)\,dz = O(R^{-2}) \to 0$.
- On γ_3 we have $\int_{\gamma_3} f(z)\,dz = -\int_{1/R}^{R} \dfrac{e^{-2it}}{t(t^2+1)}\,dt \to -\int_0^\infty \dfrac{e^{-2it}}{t(t^2+1)}\,dt$ as $R \to \infty$.
- On γ_4, since it is a clockwise semicircle tending to zero around the simple pole at $z = 0$ of $f(z)$ at which the residue equals $\lim_{z\to 0} zf(z) = 1$, by the fractional residues lemma the integral tends to $-\pi i$ as $R \to \infty$.
- So taking $R \to \infty$ yields $-\pi i e^{-3} = \int_0^\infty \dfrac{e^{3it}}{t(t^2+1)}\,dt + 0 - \int_0^\infty \dfrac{e^{-3it}}{t(t^2+1)}\,dt - \pi i$ which simplifies to $\pi i(1-e^{-3}) = \int_0^\infty \dfrac{2i\sin 3t}{t(t^2+1)}\,dt$ so that $\int_{-\infty}^{\infty} \dfrac{\sin 3t}{t(t^2+1)}\,dt = \boxed{\pi(1-e^{-3})}$.

(e) $\int_0^\infty \dfrac{\sqrt[3]{x}}{x^2+4}\,dx$.

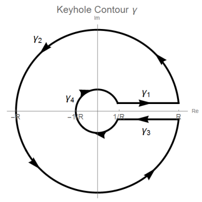
- Take integrand $f(z) = \dfrac{e^{\log(z)/3}}{z^2+4}$ where the branch cut of the logarithm is along the negative imaginary axis, and a semicircular contour that detours around 0.
- Then γ encloses the simple pole at $z = 2i$ with residue $\lim_{z\to 2i} \dfrac{z-2i}{z^2+4} e^{\log(z)/3} = \dfrac{e^{\log(2i)/3}}{4i} = \dfrac{2^{1/3}e^{i\pi/6}}{4i}$, so by the residue theorem $\int_\gamma f(z)\,dz = 2\pi i \cdot \dfrac{2^{1/3}e^{i\pi/6}}{4i} = \pi 2^{-2/3} e^{i\pi/6}$.
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{1/R}^{R} \dfrac{e^{\log(t)/3}}{t^2+1}\,dt = \int_{1/R}^{R} \dfrac{\sqrt[3]{t}}{t^2+1}\,dt \to I$ as $R \to \infty$.
- On γ_2 we have $|f(z)| = O(R^{-5/3})$ so since the arclength of γ_2 is πR we see $\int_{\gamma_2} f(z)\,dz = O(R^{-2/3}) \to 0$.
- On γ_3 we have $\int_{\gamma_3} f(z)\,dz = \int_{1/R}^{R} \dfrac{e^{\log(-t)/3}}{(-t)^2+4}\,dt = \int_{1/R}^{R} \dfrac{e^{i\pi/3}\sqrt[3]{t}}{t^2+4}\,dt \to e^{i\pi/3} I$ as $R \to \infty$.
- On γ_4 we have $|f(z)| = O(R^{-1/3})$ so since the arclength of γ_4 is π/R we see $\int_{\gamma_4} f(z)\,dz = O(R^{-4/3}) \to 0$.
- So taking $R \to \infty$ yields $\pi 2^{-2/3} e^{i\pi/6} = I + 0 - e^{i\pi/3} I + 0$ so that $I = \dfrac{\pi 2^{-2/3} e^{i\pi/6}}{1+e^{i\pi/3}} = \dfrac{\pi 2^{-2/3}}{e^{-i\pi/6}+e^{i\pi/6}} = \dfrac{\pi 2^{-2/3}}{2\cos(\pi/6)} = \boxed{\dfrac{\pi}{2^{2/3}\sqrt{3}}}$.

(f) $\int_{-\infty}^{\infty} \dfrac{x^2\cos 2x}{(x^2+1)^2}\,dx$.

- We take $f(z) = \dfrac{z^2 e^{2iz}}{(z^2+1)^2}$ and integrate around the rectangle with vertices $\pm R$ and $\pm R + iR$.
- There are poles at $z = \pm i$, but only the pole at $z = i$ lies inside the contour. The residue at $z = i$ is $\lim_{z\to i} \dfrac{d}{dz}[(z-i)^2 f(z)] = \lim_{z\to i} \left[2ze^{2iz}(z+i)^{-2} + 2iz^2e^{2iz}(z+i)^{-2} - 2z^2e^{2iz}(z+i)^{-3}\right] = -\dfrac{1}{4}e^{-2}i$, so by the residue theorem the integral on the contour is $\dfrac{1}{2}\pi e^{-2}$.
- For $z = \pm R + it$ and $iR + t$ with $-R \le t \le R$, we see that $|f(z)| = O(R^{-2})$ so since each component has length $O(R)$, the integral on each component is $O(R^{-1})$ hence tends to 0 as $R \to \infty$.
- On the real axis the integral tends to $\int_{-\infty}^{\infty} \dfrac{x^2 e^{2ix}}{(x^2+1)^2}\,dx$ as $R \to \infty$.
- Thus $\int_{-\infty}^{\infty} \dfrac{xe^{2ix}}{(x^2+1)^2}\,dx = \dfrac{1}{2}\pi e^{-2}$ so taking the real part yields $\int_{-\infty}^{\infty} \dfrac{x^2\cos 2x}{(x^2+1)^2}\,dx = \boxed{\dfrac{1}{2}\pi e^{-2}}$.

(g) $\int_0^\infty \dfrac{\ln x}{x^2+1}\,dx$.

- Take integrand $f(z) = \dfrac{\mathrm{Log}(z)^2}{z^2+1}$ and integrate around the keyhole contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:



- Then γ encloses the simple poles at $z = \pm i$ with residues $\mathrm{Res}_f(-i) = \lim_{z\to -i} \dfrac{z+i}{z^2+1}\mathrm{Log}(z)^2 = \dfrac{1}{-2i}(i\pi/2)^2$ and $\mathrm{Res}_f(i) = \lim_{z\to i} \dfrac{z-i}{z^2+1}\mathrm{Log}(z)^2 = \dfrac{1}{2i}(3i\pi/2)^2$, so by the residue theorem $\int_\gamma f(z)\,dz = 2\pi i \cdot [\mathrm{Res}_f(i) + \mathrm{Res}_f(-i)] = 2\pi^3$.
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{1/R}^R \dfrac{\mathrm{Log}(t+\epsilon i)^2}{(t+\epsilon i)^2+1}\,dt \to \int_0^\infty \dfrac{\ln(t)^2}{t^2+1}\,dt$ as $R \to \infty$.
- On γ_2 we have $|f(z)| = O(R^{-2}\ln^2 R)$ so since the arclength of γ_2 is $\le 2\pi R$ we see $\int_{\gamma_2} f(z)\,dz = O(R^{-1}\ln^2 R) \to 0$.
- On γ_3 we have $\int_{\gamma_3} f(z)\,dz = -\int_{1/R}^R \dfrac{\mathrm{Log}(t-\epsilon i)^2}{(t-\epsilon i)^2+1}\,dt \to -\int_0^\infty \dfrac{[\ln(t)+2\pi i]^2}{t^2+1}\,dt$ as $R\to\infty$.
- On γ_4 we have $|f(z)| = O(\ln^2 R)$ so since the arclength of γ_4 is $\le 2\pi/R$ we see $\int_{\gamma_4} f(z)\,dz = O(R^{-1}\ln^2 R) \to 0$.
- So taking $R\to\infty$ yields $2\pi^3 = \int_0^\infty \dfrac{\ln(t)^2}{t^2+1}\,dt - \int_0^\infty \dfrac{[\ln(t)+2\pi i]^2}{t^2+1}\,dt = 4\pi^2 \int_0^\infty \dfrac{1}{t^2+1}\,dt - 4\pi i \int_0^\infty \dfrac{\ln t}{t^2+1}\,dt$ so taking imaginary parts yields $\int_0^\infty \dfrac{\ln t}{t^2+1}\,dt = \boxed{0}$.

(h) $\int_0^\infty \dfrac{(\ln x)^2}{x^2+1}\,dx$.

- Take integrand $f(z) = \dfrac{\mathrm{Log}(z)^3}{z^2+1}$ and integrate around the keyhole contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$.
- Then γ encloses the simple poles at $z = \pm i$ with residues $\mathrm{Res}_f(-i) = \lim_{z\to -i} \dfrac{z+i}{z^2+1}\mathrm{Log}(z)^3 = \dfrac{1}{-2i}(3i\pi/2)^3$ and $\mathrm{Res}_f(i) = \lim_{z\to i} \dfrac{z-i}{z^2+1}\mathrm{Log}(z)^3 = \dfrac{1}{2i}(i\pi/2)^3$, so by the residue theorem $\int_\gamma f(z)\,dz = 2\pi i \cdot [\mathrm{Res}_f(i) + \mathrm{Res}_f(-i)] = \dfrac{13}{4} i\pi^4$.
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{1/R}^R \dfrac{\mathrm{Log}(t+\epsilon i)^2}{(t+\epsilon i)^2+1}\,dt \to \int_0^\infty \dfrac{\ln(t)^3}{t^2+1}\,dt$ as $R\to\infty$.
- On γ_2 we have $|f(z)| = O(R^{-2}\ln^3 R)$ so since the arclength of γ_2 is $\le 2\pi R$ we see $\int_{\gamma_2} f(z)\,dz = O(R^{-1}\ln^3 R) \to 0$.
- On γ_3 we have $\int_{\gamma_3} f(z)\,dz = -\int_{1/R}^R \dfrac{\mathrm{Log}(t-\epsilon i)^3}{(t-\epsilon i)^2+1}\,dt \to -\int_0^\infty \dfrac{[\ln(t)+2\pi i]^3}{t^2+1}\,dt$ as $R\to\infty$.
- On γ_4 we have $|f(z)| = O(\ln^3 R)$ so since the arclength of γ_4 is $\le 2\pi/R$ we see $\int_{\gamma_4} f(z)\,dz = O(R^{-1}\ln^3 R) \to 0$.
- So taking $R\to\infty$ yields $\dfrac{13}{4}i\pi^4 = \int_0^\infty \dfrac{\ln(t)^2}{t^2+1}\,dt - \int_0^\infty \dfrac{[\ln(t)+2\pi i]^3}{t^2+1}\,dt = 12\pi^2 \int_0^\infty \dfrac{\ln t}{t^2+1}\,dt + \left[8\pi^3 \int_0^\infty \dfrac{1}{t^2+1}\,dt - 6\pi \int_0^\infty \dfrac{(\ln t)^2}{t^2+1}\,dt\right] i$ so taking imaginary parts yields $8\pi^3 \int_0^\infty \dfrac{1}{t^2+1}\,dt - 6\pi \int_0^\infty \dfrac{(\ln t)^2}{t^2+1}\,dt = \dfrac{13}{4}\pi^4$. Since $\int_0^\infty \dfrac{1}{t^2+1}\,dt = \dfrac{\pi}{2}$ we obtain $\int_0^\infty \dfrac{(\ln t)^2}{t^2+1}\,dt = \boxed{\dfrac{1}{8}\pi^3}$.

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2. The goal of this problem is to evaluate the Fresnel integrals $\int_0^\infty \sin(x^2)\,dx$ and $\int_0^\infty \cos(x^2)\,dx$ using the Gaussian integral $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$ (see problem 4). Let $R > 0$ and let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where γ_1 is the line segment from 0 to R, γ_2 is the counterclockwise circular arc of $|z| = R$ from R to $Re^{i\pi/4}$, and γ_3 is the line segment from $Re^{i\pi/4}$ to 0.

   (a) Show that $\int_\gamma e^{-z^2}\,dz = 0$.

   - Since $f(z) = e^{-z^2}$ is entire, we have $\int_\gamma e^{-z^2}\,dz = 0$ since γ is a closed contour.

   (b) Show that $\int_{\gamma_1} e^{-z^2}\,dz \to \dfrac{1}{2}\sqrt{\pi}$ as $R \to \infty$.

   - On γ_1 we have $\int_{\gamma_1} e^{-z^2}\,dz = \int_0^R e^{-t^2}\,dt \to \int_0^\infty e^{-x^2}\,dx = \dfrac{1}{2}\sqrt{\pi}$ as $R \to \infty$.

   (c) Show that $\int_{\gamma_2} e^{-z^2}\,dz \to 0$ as $R \to \infty$. [Hint: Use the inequality $\cos 2t \geq 1 - 4t/\pi$ for $0 \leq t \leq \pi/4$.]

   - On γ_2 we have $\int_{\gamma_2} e^{-z^2}\,dz = \int_0^{\pi/4} e^{-R^2(\cos 2t + i \sin 2t)} \cdot iRe^{it}\,dt$.
   - Since $\left| e^{-R^2(\cos 2t + i\sin 2t)} \cdot iRe^{it} \right| = Re^{-R^2 \cos 2t}$, applying the inequality in the hint (which follows because $\cos 2t$ is convex and thus lies above its secant line on $[0, \pi/4]$), yields $Re^{-R^2 \cos 2t} \leq Re^{-R^2(1-4t/\pi)}$.
   - Thus the triangle inequality yields $\left| \int_{\gamma_2} e^{-z^2}\,dz \right| \leq \int_0^{\pi/4} Re^{-R^2(1-4t/\pi)}\,dt = \dfrac{\pi}{4R}(1 - e^{-R^2})$. This tends to zero as $R \to \infty$, as required.

   (d) Prove that $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \dfrac{\sqrt{2\pi}}{4}$.

   - On γ_3 we have $\int_{\gamma_3} e^{-z^2}\,dz = -\int_0^R e^{-(t \cdot e^{i\pi/4})^2} e^{i\pi/4} dt \to -e^{i\pi/4} \int_0^R e^{-it^2}\,dx = -e^{i\pi/4} \int_0^R [\cos(-t^2) - i\sin(-t^2)]\,dt$, which as $R \to \infty$ tends to $-\dfrac{\sqrt{2}}{2}[\int_0^\infty \cos(t^2)\,dt + \int_0^\infty \sin(t^2)\,dt] - \dfrac{\sqrt{2}}{2}\left[\int_0^\infty \cos(t^2)\,dt - \int_0^\infty \sin(t^2)\,dt\right]$.
   - Then since $\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}$ we see $0 = \dfrac{1}{2}\sqrt{\pi} + 0 - e^{i\pi/4} \int_0^\infty \cos(t^2)\,dt + ie^{i\pi/4} \int_0^\infty \sin(t^2)\,dt$. So this implies $\dfrac{1}{2}\sqrt{\pi} = \dfrac{\sqrt{2}}{2}[\int_0^\infty \cos(t^2)\,dt + \int_0^\infty \sin(t^2)\,dt] + \dfrac{\sqrt{2}}{2}\left[\int_0^\infty \cos(t^2)\,dt - \int_0^\infty \sin(t^2)\,dt\right]$.
   - Finally, comparing real and imaginary parts yields $\int_0^\infty \cos(t^2)\,dt + \int_0^\infty \sin(t^2)\,dt = \dfrac{\sqrt{2\pi}}{2}$ and $\int_0^\infty \cos(t^2)\,dt - \int_0^\infty \sin(t^2)\,dt = 0$, which yields $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \dfrac{\sqrt{2\pi}}{4}$, as desired.

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3. The goal of this problem is to give another another another another proof of the fundamental theorem of algebra. Let $p(z) = a_d z^d + \cdots + a_1 z + a_0$ be a polynomial with $a_d \neq 0$.

   (a) Let γ_R be the counterclockwise circle $|z| = R$. Show that $\lim_{R\to\infty} \int_{\gamma_R} \dfrac{p'(z)}{p(z)}\,dz$ is not zero, and calculate the exact value. [Hint: Show that $\dfrac{p'(Re^{it})}{p(Re^{it})}Re^{it} - d$ is $O(R^{-1})$.]

   - Parametrizing $\gamma_R(t) = Re^{it}$ for $0 \leq t \leq 2\pi$ yields $\int_{\gamma_R} \dfrac{p'(z)}{p(z)}\,dz = \int_0^{2\pi} \dfrac{da_d(Re^{it})^{d-1} + \cdots + a_1}{a_d(Re^{it})^d + \cdots + a_1(Re^{it}) + a_0} \cdot iRe^{it}\,dt = i\int_0^{2\pi} \dfrac{da_d + (d-1)a_{d-1}(Re^{it})^{-1} + \cdots + a_1(Re^{it})^{1-d}}{a_d + a_{d-1}(Re^{it})^{-1} + \cdots + a_0(Re^{it})^{-d}}\,dt$.
   - As $R \to \infty$ all of the terms after the first in the numerator and denominator tend to zero, and in fact the difference between $\dfrac{da_d}{a_d} = d$ and the integrand is $O(R^{-1})$ as $R \to \infty$, since the difference is $(Re^{it})^{-1} \dfrac{-a_{d-1} + \cdots + (1-d)a_1(Re^{it})^{2-d}}{a_d + a_{d-1}(Re^{it})^{-1} + \cdots + a_0(Re^{it})^{-d}}$ and the second term has a limit as $R \to \infty$.
   - Therefore the integrand converges uniformly to $\dfrac{da_d}{a_d} = d$, so by our results the integral converges to $i\int_0^{2\pi} d\,dt = 2\pi i d$ as $R \to \infty$.

(b) Show that $p(z)$ has d zeroes (counting multiplicities) in $\mathbb{C}$.

- By our zero-counting results, for a simple closed contour γ oriented counterclockwise, we know that $\int_\gamma \frac{p'(z)}{p(z)}\,dz$ is equal to $2\pi i$ times the number of zeroes of $p(z)$ (counting multiplicities) inside γ.
- By (a) since the integral has limit $2\pi id$, we see that $\int_{\gamma_R} \frac{p'(z)}{p(z)}\,dz = 2\pi id$ for sufficiently large R, since the value is $2\pi i$ times an integer. In particular, this means $p(z)$ has exactly d zeroes inside γ_R for sufficiently large R, which implies it has exactly d zeroes inside $\mathbb{C}$.

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4. [Challenge] Recall the definition of the gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt$, which converges for $\mathrm{Re}(\alpha) > 0$. The goal of this problem is to prove the reflection identity $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}$ for $0 < \mathrm{Re}(\alpha) < 1$.

(a) If $0 < \alpha < 1$, show that $\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty \int_0^\infty (s/t)^\alpha e^{-(s+t)}\,s^{-1}dt\,ds$.

- Per the definition, we have $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1}e^{-s}ds$ and $\Gamma(1-\alpha) = \int_0^\infty t^{(1-\alpha)-1}e^{-t}\,dt = \int_0^\infty t^{-\alpha}e^{-t}\,dt$.
- So the product is $\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty \int_0^\infty s^{\alpha-1}t^{-\alpha}e^{-(s+t)}\,dt\,ds = \int_0^\infty \int_0^\infty (s/t)^\alpha e^{-(s+t)}\,s^{-1}dt\,ds$.

(b) If $0 < \alpha < 1$, show that $\int_0^\infty \int_0^\infty (s/t)^\alpha e^{-(s+t)}\,s^{-1}dt\,ds = \int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx$. [Hint: Make a change of variables $r = s+t$ and $x = s/t$ and verify that $x^{-1}(1+x)^{-1}\,dr\,dx = s^{-1}\,dt\,ds$.]

- We have $r = s+t$ and $x = s/t$, so the integration range $0 \le s,t$ becomes $0 \le r,x$.
- For the Jacobian, with $r = s+t$ and $x = s/t$ we get $\left|\frac{\partial(r,x)}{\partial(s,t)}\right| = \left|\begin{matrix} \partial r/\partial s & \partial r/\partial t \\ \partial x/\partial s & \partial x/\partial t \end{matrix}\right| = \left|\begin{matrix} 1 & 1 \\ 1/t & -s/t^2 \end{matrix}\right| = (s+t)/t^2 = x(1+x)/s$, meaning that $x^{-1}(1+x)^{-1}dr\,dx = s^{-1}\,dt\,ds$.
- Changing coordinates and then separating variables yields $\int_0^\infty \int_0^\infty (s/t)^\alpha e^{-(s+t)}\,s^{-1}dt\,ds = \int_0^\infty \int_0^\infty x^\alpha e^{-r}x^{-1}(1+x)^{-1}\,dr\,dx = [\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx][\int_0^\infty e^{-r}\,dr] = \int_0^\infty \frac{x^{\alpha-1}}{x+1}dx$, as claimed.

(c) If $0 < \alpha < 1$, show that $\int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx = \frac{\pi}{\sin(\pi\alpha)}$. [Hint: Use the keyhole contour.]

- We integrate $f(z) = \frac{z^{\alpha-1}}{z+1} = \frac{e^{(\alpha-1)\mathrm{Log} z}}{z+1}$ on the keyhole contour.
- The function $f(z)$ has a branch cut discontinuity on $[0,\infty)$ and a simple pole at $z = -1$ with residue $\lim_{z\to-1}(z+1)f(z) = e^{(\alpha-1)\mathrm{Log}(-1)} = e^{i\pi(\alpha-1)} = -e^{i\pi\alpha}$, so by the residue theorem we see $\int_\gamma f(z)\,dz = -2\pi i e^{i\pi\alpha}$.
- Now we calculate the integral on each piece.
- On γ_1 we have $\int_{\gamma_1} f(z)\,dz = \int_{1/R'}^{R''} \frac{(t+\epsilon i)^{\alpha-1}}{(t+\epsilon i)+1}\,dt$, which tends to $\int_0^\infty \frac{t^{\alpha-1}}{t+1}\,dt$ as $R \to \infty$.
- On γ_2 we have $|f(z)| = \left|\frac{e^{(\alpha-1)(\ln R+it)}}{Re^{it}+1}\right| \le \frac{R^{\alpha-1}}{R-1} = O(R^{\alpha-2})$ so since the arclength of γ_2 is $< 2\pi R$ we see that $\int_{\gamma_2} f(z)\,dz = O(R^{\alpha-1}) \to 0$ as $R \to \infty$ since $\alpha < 1$.
- On γ_3, noting the reversed orientation, we have $\int_{\gamma_1} f(z)\,dz = -\int_{1/R'}^{R''} \frac{(t-\epsilon i)^{\alpha-1}}{(t-\epsilon i)+1}\,dt$, which because of the selection of the complex logarithm has the argument of the numerator increased by $2\pi i$ relative to the original integral on γ_1. Thus this integral tends to $-e^{2\pi i(\alpha-1)}\int_0^\infty \frac{t^{\alpha-1}}{t+1}\,dt$ as $R \to \infty$.
- On γ_4 we have $|f(z)| = \left|\frac{e^{(\alpha-1)(-\ln R+it)}}{e^{it}/R+1}\right| \le \frac{R^{1-\alpha}}{1-1/R} = O(R^{1-\alpha})$ so since the arclength of γ_2 is $< 2\pi/R$ we see that $\int_{\gamma_2} f(z)\,dz = O(R^{-\alpha}) \to 0$ as $R \to \infty$ since $\alpha > 0$.
- So taking $R \to \infty$ and putting all of this together yields $-2\pi i e^{i\pi\alpha} = (1 + e^{2\pi i(\alpha-1)})\int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx$, which produces $\int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx = \frac{-2\pi i e^{\pi i\alpha}}{1-e^{2\pi i\alpha}} = \frac{\pi}{(-e^{-\pi i\alpha}+e^{\pi i\alpha})/(2i)} = \frac{\pi}{\sin(\pi\alpha)}$, as claimed.

(d) Show that $\Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{\sin(\pi\alpha)}$ for all $\alpha \in \mathbb{C}$ with $0 < \mathrm{Re}(\alpha) < 1$.

- As shown in (a)-(c), the two sides are equal on the real interval $(0, 1)$. But since both sides are holomorphic on the region $0 < \mathrm{Re}(\alpha) < 1$ and they are equal on a set with an accumulation point, they must be equal everywhere in the region.

(e) Compute $\Gamma(1/2)$ and use the result to evaluate the Gaussian integral $\int_0^\infty e^{-x^2}\,dx = \dfrac{1}{2}\sqrt{\pi}$.

- Setting $\alpha = 1/2$ yields $\Gamma(1/2)^2 = \dfrac{\pi}{\sin(\pi/2)} = \pi$. Since the integral for the gamma function is clearly positive, taking the square root yields $\Gamma(1/2) = \sqrt{\pi}$.
- For the second part we have by definition $\Gamma(1/2) = \int_0^\infty t^{-1/2}e^{-t}\,dt$.
- Substituting $t = x^2$ so that $dt = 2x\,dx$, and with $t = 0$ corresponding to $x = 0$ and $t = \infty$ corresponding to $x = \infty$, produces $\Gamma(1/2) = \int_0^\infty x^{-1}e^{-x^2} \cdot 2x\,dx = 2\int_0^\infty e^{-x^2}$. So this immediately gives $\int_0^\infty e^{-x^2}\,dx = \dfrac{1}{2}\sqrt{\pi}$ as claimed.
