E. Dummit's Math 4555 ~ Complex Analysis, Fall 2025 ~ Homework 1, due Fri Sep 12th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Express the following complex numbers in rectangular $a + bi$ form:
   (a) $(3+i) - (4-2i)(1-i)$.
   (b) $(4+3i)/(5-i)$.
   (c) $4e^{7i\pi/6}$.
   (d) $e^{i\pi/4} + 3e^{3i\pi/4}$.
   (e) $e^{2025i\pi/3}$.
   (f) $(1+i)^{2024}$.

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2. Express the following complex numbers in exponential $re^{i\theta}$ form:
   (a) $3i$.
   (b) $-2 - 2i\sqrt{3}$.
   (c) $-1 + i$.
   (d) $\pi + ei$.

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3. Find all $z \in \mathbb{C}$ satisfying the following equations:
   (a) $z^2 + 2z + 2 = 0$.
   (b) $z^2 = 3 + 4i$.
   (c) $z^2 - (2-i)z - 2i = 0$.
   (d) $z^3 = -1$.
   (e) $z^8 = 1$.
   (f) $z^4 = 3$.
   (g) $e^z = 1$.
   (h) $e^z = 1 - i\sqrt{3}$.

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4. Plot each of the given regions in the complex plane (you may want to use a computer). For each region, identify whether it is (i) open, (ii) closed, (iii) connected, and (iv) bounded.
   (a) $|z - 1| < 1$.
   (b) $1 \le |z| \le 3$.
   (c) $|z| > |z - 1|$.
   (d) $0 < \mathrm{Im}(z) \le 1$.
   (e) $\mathrm{Re(z)} \le \mathrm{Im(z)}$.
   (f) $\mathrm{Re}(z) \cdot \mathrm{Im}(z) > 1$.
   (g) $|z(z-2)| \le 1$.
   (h) $|z(z-4)| \le 1$.

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5. Compute the following complex limits or show they do not exist:
   (a) $\lim_{z \to 1} \dfrac{z^3 - 1}{z - 1}$.
   (b) $\lim_{z \to i} \dfrac{z^3 - 1}{z - 1}$.
   (c) $\lim_{z \to 0} \dfrac{z}{|z|}$.
   (d) $\lim_{z \to 0} \dfrac{z^3}{|z|^2}$.
   (e) $\lim_{z \to 1} \dfrac{z^2 - 1}{z^2 + z - 2\overline{z}}$.
   (f) $\lim_{z \to 0} \dfrac{z^{720}}{|z|^{720}}$.
   (g) $\lim_{z \to 0} \dfrac{\mathrm{Re}(z) \cdot \mathrm{Im}(z)^2}{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^4}$. [Hint: Try the path $z = t^2 + it$ as $t \to 0$.]

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

6. The goal of this problem is to illustrate some uses of complex exponentials for trigonometry. You may assume in this problem that indefinite integrals involving complex parameters behave the same way as if the parameters were real (we will later prove this), and you may use Euler's identity $e^{ix} = \cos x + i \sin x$.
   (a) If x is real, show that $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$.
   (b) Compute $\int \cos^4 x \, dx$ and $\displaystyle\int \sin^4 x \, dx$. [Hint: Use (a).]
   (c) Compute $\displaystyle\int e^{ax} \cos bx \, dx$ and $\displaystyle\int e^{ax} \sin bx \, dx$. [Hint: Take real and imaginary parts of $\displaystyle\int e^{(a+bi)x} \, dx$.]
   (d) Prove that $1 + e^{ix} + e^{2ix} + \cdots + e^{inx} = \dfrac{e^{(n+1)ix} - 1}{e^{ix} - 1} = e^{(n/2)ix} \dfrac{\sin[\frac{n+1}{2}x]}{\sin(x/2)}$ for any positive integer n and any $0 < x < 2\pi$.
   (e) Deduce that $1 + \cos x + \cos 2x + \cdots + \cos nx = \dfrac{\sin[\frac{n+1}{2}x] \cos[\frac{n}{2}x]}{\sin(x/2)}$ and $\sin x + \sin 2x + \cdots + \sin nx = \dfrac{\sin[\frac{n+1}{2}x] \sin[\frac{n}{2}x]}{\sin(x/2)}$ for any positive integer n and any $0 < x < 2\pi$.

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7. The real numbers are an example of an ordered field, which is a field containing a subset P (the "positive" elements) such that (i) P is closed under addition, (ii) P is closed under multiplication, and (iii) every nonzero element of the field is either in P or its additive inverse is in P but not both. Prove that $\mathbb{C}$ is not an ordered field for any possible choice of P. [Hint: Consider i.]

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8. The goal of this problem is to illustrate one of the original historical applications of the complex numbers: that of solving the cubic equation.

   (a) Suppose that $z^3 + az^2 + bz + c = 0$. Show that $t = z + a/3$ has $t^3 + pt + q = 0$ where $p = b - a^2/3$ and $q = (2/27)a^3 - ab/3 + c$. Thus, it suffices to solve cubics of the form $t^3 + pt + q = 0$.

   (b) Suppose that $t^3 + pt + q = 0$. Define new variables x and y such that $x + y = t$ and $3xy = -p$. Show that $x^3 + y^3 = -q$ and then solve for x^3 and y^3.

   (c) Conclude that the solutions to the cubic $t^3 + pt + q = 0$ are the three numbers of the form $t = A + B$, with $A = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ and $B = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, where the cube roots are selected so that $AB = -p/3$. These formulas are known as Cardano's formulas.

   (d) Verify that the cubic $f(t) = t^3 - 15t - 4$ has three real roots and that they are 4 and $-2 \pm \sqrt{3}$.

   (e) Use Cardano's formulas to find the roots of $f(t) = t^3 - 15t - 4$, and then show that they do simplify to yield the same answers in (d) using the calculation $(2 + i)^3 = 2 + 11i$.

Remark: The calculation in (e) was performed by Bombelli in 1572. This rather perplexing appearance of square roots of negative numbers in the formulas for real solutions to cubic equations was the original impetus that led to the development and acceptance of complex numbers in mathematics (although unsurprisingly, it did take a while!).

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9. [Challenge] The goal of this problem is to justify the remark following problem 8 by showing that for *any* cubic polynomial whose roots are real, Cardano's formulas necessarily involve non-real radicals. Suppose $f(z) = z^3 + az^2 + bz + c = (z - r_1)(z - r_2)(z - r_3)$ is a cubic polynomial with real coefficients $a, b, c \in \mathbb{R}$ and complex roots $r_1, r_2, r_3 \in \mathbb{C}$. Define its discriminant as $\Delta = (r_1 - r_2)^2(r_1 - r_3)^2(r_2 - r_3)^2$ and let $p = b - a^2/3$ and $q = (2/27)a^3 - ab/3 + c$.

   (a) Show that $\Delta = 0$ when f has a repeated root, that $\Delta > 0$ when f has three distinct real roots, and that $\Delta < 0$ when f has a pair of nonreal complex-conjugate roots.

   (b) Show that the discriminant of $f(t)$ is the same as that of $g(t) = t^3 + pt + q$ where $p = b - a^2/3$ and $q = (2/27)a^3 - ab/3 + c$.

   (c) Show that the discriminant of $f(t)$ equals $-27p^3 - 4q^2$. [Hint: Δ is a symmetric function in r_1, r_2, r_3 and must therefore be a polynomial in the three functions $r_1 + r_2 + r_3 = 0$, $r_1r_2 + r_1r_3 + r_2r_3 = p$, and $r_1r_2r_3 = -q$. Since it has degree 6, it must be of the form $Xp^3 + Yq^2$. Plug in values for r_1, r_2, r_3 subject to $r_1 + r_2 + r_3 = 0$ to find the coefficients.]

   (d) Show that if f has three distinct real roots, then Cardano's formulas require computing cube roots of non-real complex numbers.
