Lecture: Monday-Wednesday-Thursday 4:35pm–5:40pm, Snell Library 049.

Instructor: Evan Dummit, Lake Hall 571, edummit@northeastern.edu.

Office Hours: Mon/Wed/Thu 3:00pm-4:00pm, or by appointment, 571 Lake Hall.

Course Webpage: https://web.northeastern.edu/dummit/teaching_fa24_7315.html.

Course Textbooks: The instructor will write course notes in lieu of an official textbook. The course will generally begin by following Marcus's "Number Fields", with other good references on these topics being Shafarevich and Borevich's "Number Theory", Narkiewicz's "Elementary and Analytic Theory of Algebraic Numbers", and for the more advanced with algebraic-geometry background, Neukirch's "Algebraic Number Theory".

Expected Background: There are no formal prerequisites, but students should have comfort with algebra at the beginning graduate level (Math 5111 or 5112). I will freely refer to some results from commutative algebra, and potentially complex analysis, but the goal is to make the course as self-contained as possible.

Course Overview: Classical elementary number theory consists of studying the properties of arithmetic in $\mathbb{Q}$ and its associated ring of integers $\mathbb{Z}$ (e.g., prime factorization, Fermat's little theorem, quadratic reciprocity, the prime number theorem, etc.). Algebraic number theory extends the study of these topics to the larger class of number fields, which are the finite algebraic extensions $K/\mathbb{Q}$, using the tools of modern algebra. A broad theme is to study the presence (and failure) of unique factorization, with a particular highlight being the study of prime ideals and their splitting in extensions using Galois theory. Analytic tools such as the zeta function and more general L-functions also enter naturally into the discussion of questions about the distribution of primes. By combining all of these approaches, our aim is to extend many classical results of number theory into more general settings.

Course Topics: We begin by motivating the general study of unique factorization in number fields using some classical problems in number theory (e.g., Fermat's equation). We then establish a number of basic properties of number fields and their associated rings of integers using algebra: we review Dedekind domains, then discuss traces, norms, discriminants, unique factorization of ideals, ramification, discriminants, and the ideal class group, taking efforts to illustrate these results with calculations from accessible classes of examples such as quadratic, cubic, and cyclotomic fields. We then bring Galois theory into the discussion and study the Galois action on ideals and primes: we construct the decomposition and inertia subgroups and study the action of Frobenius. Finally, we bring in additional tools from geometry (lattices and the geometry of numbers) and analysis (the zeta function and L-functions) to extend our results further: we use Minkowski's theorem to establish the finiteness of the ideal class group and to prove Dirichlet's Unit Theorem, and we establish the analytic class number formula and study the distribution of primes.
Time permitting, we may also discuss other topics according to student and instructor interest, such as p-adic valuations and completions, local fields, ray class fields, an introduction to class field theory, or other related topics.

Grades: Grades will be based on attendance and participation (50%), and on occasional homework assignments (50%), which will be collaborative and may involve roundtable discussions of the solutions. Depending on student and instructor interest, there may also be student presentations on course-related topics during the semester.

Course Schedule: The course is tentatively organized into six sections, of which likely 4-5 will be covered:

Algebraic Number Fields (3 weeks): Algebraic integers, number fields and their rings of integers, cyclotomic fields, trace and norm, discriminants

Prime Decomposition (5 weeks): Dedekind domains, unique factorization of ideals, splitting of primes in extensions, the Galois action on prime ideals, decomposition and inertia subgroups

Geometry of Numbers (3 weeks): the ideal class group, geometry of numbers and the Minkowski bound, computing class groups, Dirichlet's unit theorem

Analytic Number Theory (3 weeks): Zeta functions, L-functions, the analytic class number formula, Chebotarev's theorem and the distribution of primes

Valuations and Local Fields (3 weeks): p-adic valuations and completions, the field $\mathbb{Q}_p$ and its extensions, number theory in local fields

Introduction to Class Field Theory (3 weeks): abelian extensions, the Kronecker-Weber theorem, ray class fields, the Hilbert class field, the correspondence between unramified abelian extensions and subgroups of the class group.

Attendance Policy: It is expected that you will attend every class (see above under "Grades"), but I am willing to work with you if you will have to miss some of the lectures. Since this is a graduate-level topics course, it is very easy to get lost even if you only miss one day.

If you will be absent from a class activity due to a religious observance or practice, or for participation in a university-sanctioned event (e.g., university athletics), it is your responsibility to inform the instructor during the first week of class and provide appropriate documentation if required.

External Resources Policy: If you use **any** external resources (e.g., wikipedia, stackexchange, other books beyond the course text or notes, other people, etc.) you must say **what results you are citing and where they are from**. If you happen to find a solution to an assigned problem online or elsewhere, it is plagiarism to present it as your own work without attribution of its source.

Use of generative AI / large language models (e.g., ChatGPT, Copilot) or similar technology, is **expressly prohibited in this course. Submitting answers generated by such models constitutes plagiarism and will receive an automatic zero on the assignment**.

Statement on Academic Integrity: A commitment to the principles of academic integrity is essential to the mission of Northeastern University. Academic dishonesty violates the most fundamental values of an intellectual community and undermines the achievements of the entire University. Violations of academic integrity include (but are not limited to) cheating on assignments or exams, fabrication or misrepresentation of data or other work, plagiarism, unauthorized collaboration, and facilitation of others' dishonesty. Possible sanctions include (but are not limited to) warnings, grade penalties, course failure, suspension, and expulsion.

Statement on Accommodations: Any student with a disability is encouraged to meet with or otherwise contact the instructor during the first week of classes to discuss accommodations. The student must bring a current Memorandum of Accommodations from the Office of Student Disability Services.

Statement on Inclusivity: Faculty are encouraged to address students by their preferred name and gender pronoun. If you would like to be addressed using a specific name or pronoun, please let your instructor know.

Statement on Evaluations: Students are requested to complete the TRACE evaluations at the end of the course.

Miscellaneous Disclaimer: The instructor reserves the right to change course policies, including the evaluation scheme of the course. Notice will be given in the event of any substantial changes.