

# Math 1465: Intensive Mathematical Reasoning

Midterm 2, Form E (Instructor: Dummit)

November 18th, 2024

NAME (please print legibly): \_\_\_\_\_

Your University ID Number: \_\_\_\_\_

- Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!
- You may appeal to results covered at any point in the course, but please make clear what results you are using. **Box** all final numerical answers.
- In problems with multiple parts, you may use the results of previous parts in later parts, even if you did not solve the earlier parts correctly.
- You are responsible for checking that this exam has all 8 pages.
- You are allowed a 1-page note sheet and a calculator. Time limit: **65 minutes**.

## Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: \_\_\_\_\_

QUESTION	VALUE	SCORE
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	15	
TOTAL	80	

1. (15 points) Solve the following (a correct answer automatically receives full credit, incorrect answers with relevant work may receive partial credit):

(a) Find the multiplicative inverse of the residue class  $\bar{5}$  modulo 31.

(b) Give an example of a relation  $R$  on the set  $\{1, 2, 3\}$  that is reflexive but not symmetric.

(c) Suppose  $R$  is the equivalence relation on  $\{1, 2, 3, 4, 5\}$  whose equivalence classes are  $\{1, 5\}$ ,  $\{2\}$ , and  $\{3, 4\}$ . List all of the ordered pairs in  $R$ .

(d) Give an example of a function  $f : \{1, 2, 3, 4\} \rightarrow \{a, b, c\}$  that is onto but not one-to-one.

(e) Find a formula for the inverse of the function  $g(x) = 5x - 3$  from  $\mathbb{Q} \rightarrow \mathbb{Q}$ .

**2. (10 points)** Let  $A = \{1, 2, 3\}$  and let  $R$  be the relation on  $A$  given by  $R = \{(1, 1), (1, 2), (2, 1), (3, 3)\}$ . Answer each of the following (give a brief justification where appropriate):

(a) Determine whether  $R$  is reflexive.

(b) Determine whether  $R$  is symmetric.

(c) Determine whether  $R$  is transitive.

(d) Determine whether  $R$  is an equivalence relation on  $A$ .

(e) Determine whether  $R$  is antisymmetric.

(f) Determine whether  $R$  is a partial ordering on  $A$ .

(g) Determine whether  $R$  is a function from  $A$  to  $A$ .

**3. (10 points)** Let  $A = \mathbb{Z}/7\mathbb{Z}$ , the set of residue classes modulo 7, and let  $R$  be the relation on  $A$  defined by

$$\bar{a} R \bar{b} \text{ if and only if } a^2 \equiv b^2 \pmod{7}.$$

(a) Prove that  $R$  is an equivalence relation.

(b) Identify all of the equivalence classes of  $R$ .

4. (10 points) Suppose that  $f : A \rightarrow B$  is a one-to-one function and  $S$  is a partial ordering on  $B$ . Prove that the relation  $R : A \rightarrow A$  given by  $R = \{(a, b) \in A \times A : (f(a), f(b)) \in S\}$  is a partial ordering on  $A$ .

**5. (10 points)** Suppose that  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is a function such that  $f(f(n)) = -n$  for all  $n \in \mathbb{Z}$ . Prove that  $f$  is one-to-one and onto. [Hint: What is  $f(f(f(f(n))))$ ?]

**6. (10 points)** Solve the following:

(a) Prove there is a bijection between the set  $S$  of perfect squares and the set  $P$  of positive prime numbers.

(b) Suppose that  $f : \mathbb{R} \rightarrow \mathbb{Z}$  is onto. Show that  $f$  cannot be one-to-one.

7. (15 points) Circle the correct response for each of the following (no work is required, and there is no partial credit or penalty for incorrect answers):

(a) Which one of the following sets is uncountable?

- (i) The set of real numbers greater than 1.
  - (ii) The set of subsets of  $\{1, 2, 3, 4, 5, 6, \dots, 2024\}$ .
  - (iii) The set of rational numbers.
  - (iv) The set of even integers.
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(b) Identify each statement as true or false (each answer is worth independent points):

**True**   **False**   Modulo 12, it is true that  $\bar{4} \cdot \bar{6} = \bar{5} + \bar{7}$ .

**True**   **False**   Every nonzero residue class modulo 11 has a multiplicative inverse.

**True**   **False**   There are exactly 2 different equivalence relations on  $\{1, 2, 3, 4, 5\}$ .

**True**   **False**   The divisibility relation is a partial ordering on  $\{-2, -1, 1, 2\}$ .

**True**   **False**   The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^3$  is one-to-one.

**True**   **False**   The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^3$  is onto.

**True**   **False**   If  $f : A \rightarrow B$  is one-to-one, then  $f$  is a bijection between  $A$  and  $B$ .

**True**   **False**   If  $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$  is onto, then  $f$  is a bijection.

**True**   **False**   There exists a bijection  $g : \mathbb{R} \rightarrow \mathbb{Q}$ .

**True**   **False**   There exists a bijection  $h : \mathbb{Z} \rightarrow \mathbb{Q}$ .

**True**   **False**   The power set of any countable set is countable.

**True**   **False**   Every subset of an uncountable set is uncountable.

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