Math 1465: Intensive Mathematical Reasoning

Midterm 2, Form D (Instructor: Dummit)

November 18th, 2024

NAME (please print legibly): ____________________

Your University ID Number: ____________________

- **Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!**
- **You may appeal to results covered at any point in the course, but please make clear what results you are using. Box all final numerical answers.**
- **In problems with multiple parts, you may use the results of previous parts in later parts, even if you did not solve the earlier parts correctly.**
- **You are responsible for checking that this exam has all 8 pages.**
- **You are allowed a 1-page note sheet and a calculator. Time limit: 65 minutes.**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ____________________

QUESTION	VALUE	SCORE
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	15	
TOTAL	80	

1. (15 points) Solve the following (a correct answer automatically receives full credit, incorrect answers with relevant work may receive partial credit):

(a) Find the multiplicative inverse of the residue class $\overline{6}$ modulo 31.

(b) Give an example of a relation R on the set $\{1, 2, 3\}$ that is reflexive but not symmetric.

(c) Suppose R is the equivalence relation on $\{1, 2, 3, 4, 5\}$ whose equivalence classes are $\{1, 5\}$, $\{2\}$, and $\{3, 4\}$. List all of the ordered pairs in R.

(d) Give an example of a function $f : \{1, 2, 3, 4\} \to \{a, b, c\}$ that is onto but not one-to-one.

(e) Find a formula for the inverse of the function $g(x) = 7x - 4$ from $\mathbb{Q} \to \mathbb{Q}$.

2. (10 points) Let $A = \{1, 2, 3\}$ and let R be the relation on A given by $R = \{(1, 2), (2, 1), (3, 3)\}$. Answer each of the following (give a brief justification where appropriate):

(a) Determine whether R is reflexive.

(b) Determine whether R is symmetric.

(c) Determine whether R is transitive.

(d) Determine whether R is an equivalence relation on A.

(e) Determine whether R is antisymmetric.

(f) Determine whether R is a partial ordering on A.

(g) Determine whether R is a function from A to A.

3. (10 points) Let $A = \mathbb{Z}/7\mathbb{Z}$, the set of residue classes modulo 7, and let R be the relation on A defined by

$$\overline{a}\ R\ \overline{b} \text{ if and only if } a^2 \equiv b^2 \pmod 7.$$

(a) Prove that R is an equivalence relation.

(b) Identify all of the equivalence classes of R.

4. (10 points) Suppose that $f : A \to B$ is a one-to-one function and S is a partial ordering on B. Prove that the relation $R : A \to A$ given by $R = \{(a, b) \in A \times A : (f(a), f(b)) \in S\}$ is a partial ordering on A.

5. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is a function such that $f(f(n)) = -n$ for all $n \in \mathbb{Z}$. Prove that f is one-to-one and onto. [Hint: What is $f(f(f(f(n))))$?]

6. (10 points) Solve the following:

(a) Prove there is a bijection between the set E of even integers and the set P of positive prime numbers.

(b) Suppose that $f : \mathbb{R} \to \mathbb{Z}$ is onto. Show that f cannot be one-to-one.

7. (15 points) Circle the correct response for each of the following (no work is required, and there is no partial credit or penalty for incorrect answers):

(a) Which one of the following sets is uncountable?

(i) The set of rational numbers.

(ii) The set of real numbers greater than 1.

(iii) The set of odd integers.

(iv) The set of subsets of $\{1, 2, 3, 4, 5, 6, \ldots, 2024\}$.

(b) Identify each statement as true or false (each answer is worth independent points):

True **False** Modulo 11, it is true that $\overline{4} \cdot \overline{6} = \overline{5} + \overline{8}$.

True **False** Every nonzero residue class modulo 13 has a multiplicative inverse.

True **False** There are exactly 2 different equivalence relations on $\{1, 2, 3, 4, 5\}$.

True **False** The divisibility relation is a partial ordering on $\{-10, -1, 1, 10\}$.

True **False** The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^4$ is one-to-one.

True **False** The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^4$ is onto.

True **False** If $f : A \to B$ is onto, then f is a bijection between A and B.

True **False** If $f : \{1, 2, 3\} \to \{a, b, c\}$ is one-to-one, then f is a bijection.

True **False** There exists a bijection $g : \mathbb{R} \to \mathbb{Q}$.

True **False** There exists a bijection $h : \mathbb{Z} \to \mathbb{Q}$.

True **False** The power set of any countable set is countable.

True **False** Every subset of an uncountable set is uncountable.
