1. We construct a truth table for each statement that shows they are equivalent:

P	Q	$P \downarrow P$	$Q \downarrow Q$	$(P \downarrow P) \downarrow (Q \downarrow Q)$	$P \wedge Q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	F	F

This result can also be shown using equivalences:

$$\begin{aligned} (P \downarrow P) \downarrow (Q \downarrow Q) &= [\neg(P \vee P)] \downarrow [\neg(Q \vee Q)] && \text{(definition of } \downarrow) \\ &= (\neg P) \downarrow (\neg Q) && \text{(absorption)} \\ &= \neg[(\neg P) \vee (\neg Q)] && \text{(definition of } \downarrow) \\ &= [\neg\neg P] \wedge [\neg\neg Q] && \text{(de Morgan)} \\ &= P \wedge Q && \text{(double negative)} \end{aligned}$$

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2. This is an if-and-only-if statement so we must prove both directions or use a chain of equivalences.

- Solution 1: First, suppose $A \subseteq B$: we must show that $B^c \subseteq A^c$. Notice that $A \subseteq B$ means that if $x \in A$ then $x \in B$. Taking the contrapositive of this statement shows that if $x \notin B$ then $x \notin A$. Equivalently, this says $x \in B^c$ implies $x \in A^c$ which means precisely that $B^c \subseteq A^c$, as desired.
- Now suppose $B^c \subseteq A^c$: we must show that $A \subseteq B$. The hypothesis says that $x \in B^c$ implies $x \in A^c$, which equivalently says if $x \notin B$ then $x \notin A$. Taking the contrapositive yields the statement that if $x \in A$ then $x \in B$, which implies $A \subseteq B$ as desired.
- Solution 2: We use a chain of equivalences: $A \subseteq B \iff x \in A$ implies $x \in B \iff x \notin B$ implies $x \notin A \iff x \in B^c$ implies $x \in A^c \iff B^c \subseteq A^c$.

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3. Each part was worth 3 points.

(a) We want a prime number n such that $n+8$ is not prime. Some examples of such n are $n = 2, 7, 13, 19$.

(b) We want a set A such that $A \subseteq \{1, 2, 3, 4, 5\}$ and $A \cap \{3, 4, 5\}$ is empty. There are several choices: $A = \{1, 2\}$, $A = \{1\}$, $A = \{2\}$, or $A = \emptyset$.

(c) We want a positive integer n that is not prime but doesn't have two different prime factors. Some examples of such n are $n = 1$ (no prime factors) or n equal to a prime power such as 4, 8, 9, 16, or 25.

(d) In words this statement says "for all real x there exists a real y such that $xy = 1$". For most values of x there is such a value of y, namely, $y = 1/x$. But for $x = 0$, there exists no y with $xy = 1$ because xy is always 0 no matter what y is. Thus, $x = 0$ is a counterexample (in fact, the only one).

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4. We use (strong) induction on n.

- For the base cases we take $n = 0$ and $n = 1$: we have $b_0 = 2 = 3^0 + 1$ and $b_1 = 4 = 3^1 + 1$ as required.
- For the inductive step, suppose that $b_{n-1} = 3^{n-1} + 1$ and $b_{n-2} = 3^{n-2} + 1$. Then by definition we see $b_n = 4b_{n-1} - 3b_{n-2} = 4(3^{n-1} + 1) - 3(3^{n-2} + 1) = 4 \cdot 3^{n-1} + 4 - 3 \cdot 3^{n-2} - 3 = 3 \cdot 3^{n-1} + 1 = 3^n + 1$, as required.
- Therefore, the result holds for all $n \geq 0$ by induction.
- Remark: Common errors included starting with the wrong base case, failing to do 2 base cases, and writing the steps in the proof of the inductive step in the wrong order.

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5. Part (a) was worth 5 points while (b) and (c) were worth 4 points.

   (a) We use the Euclidean algorithm:

$$\begin{aligned} 460 &= 10 \cdot 44 + 20 \\ 44 &= 2 \cdot 20 + 4 \\ 20 &= 5 \cdot 4 \end{aligned}$$

   The last nonzero remainder is 4, so the greatest common divisor is $\boxed{4}$.

   (b) We have $\text{lcm}(a, b) = ab/\gcd(a, b) = \boxed{460 \cdot 44/4 = 5060}$.

   (c) We use the extended Euclidean algorithm to solve for the remainders one at a time. This yields

$$\begin{aligned} 20 &= 460 - 10 \cdot 44 \\ 4 &= 44 - 2 \cdot 20 = 44 - 2 \cdot (460 - 10 \cdot 44) = -2 \cdot 460 + 21 \cdot 44 \end{aligned}$$

   Therefore, we can take $x = \boxed{-2}$ and $y = \boxed{21}$.

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6. This is an if-and-only-if statement so we must prove both directions or use a chain of equivalences.

   - Solution 1: First suppose that $\gcd(a, b) = a$. Then since the gcd of two numbers is a divisor of both, that means $a|a$ and $a|b$, so in particular $a|b$.
   - Conversely, suppose $a|b$. Then a is a common divisor of a and b, and also any other common divisor d must divide a hence $d \leq a$: therefore, a is the greatest common divisor of a and b.
   - Solution 2: If $\gcd(a, b) = a$ then as in Solution 1 that means $a|b$. Conversely, if $a|b$ then $b = ka$ for some integer k, and then $\gcd(a, b) = \gcd(a, ka) = a \cdot \gcd(1, k) = a$ by gcd properties.
   - Solution 3: Suppose a and b have prime factorizations $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $b = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$. Then $\gcd(a, b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \cdots p_k^{\min(a_k,b_k)}$ which equals a if and only if $a_i = \min(a_i, b_i)$ for each i. But that is true if and only if $a_i \leq b_i$, which is in turn true if and only if $a|b$.

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7. Parts (a)-(c) were each worth 3 points, while each item in (d) was worth 1 point.

   (a) Symbolically, the given statement is $\forall x \in A\, \exists y \in B,\ (x + y) \in C$. Negating flips each quantifier to the other type (but preserves their order), so the negation is $\boxed{\text{(iv) } \exists x \in A\, \forall y \in B,\ (x + y) \notin C}$.

   (b) The mistake is that $\boxed{\text{(iii) The proof only establishes one implication of the biconditional.}}$ The statement of the proposition is an if-and-only-if statement, but only the implication "if m is odd then m^2 is odd" is shown (the converse is not). The other three choices do not apply: it is valid to start showing the conditional by assuming m is even, there is no justification needed for why $m = 2k$ (it is the definition of an even number), and the proof does not need to start by assuming m^2 is even (that would be for a proof of the converse).

   (c) The mistake is that $\boxed{\text{(i) The argument starts with the wrong base case.}}$ The proposition claims the result for all positive integers n, but the base case starts with $n = 2$ instead of $n = 1$. The other three choices do not apply: the argument does not need two base cases, the result in the inductive step is correctly shown as $1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = (n + 1)^2$, and the inductive step should not start by assuming that statement.

   (d) In order, the responses are

      i. $\boxed{\text{True}}$, since $(F \wedge \neg T) \Rightarrow (\neg F)$ resolves to $F \Rightarrow T$ which is true.

      ii. $\boxed{\text{False}}$ because the set $\{1\}$ is not an element of A (the number 1 is, but $1 \neq \{1\}$).

      iii. $\boxed{\text{True}}$ because the empty set is a subset of every set, including A.

      iv. $\boxed{\text{True}}$ by the formula for the gcd in terms of a prime factorization, the exponent of each prime in the gcd is the minimum of the corresponding exponents in the given integers.

      v. $\boxed{\text{False}}$ because having $2^a = 3^b$ for positive integers a, b would contradict the uniqueness of prime factorizations.

      vi. $\boxed{\text{False}}$, as proven by Euclid and shown in class, there are infinitely many prime numbers.
