1. We construct a truth table for each statement that shows they are equivalent:

P	Q	$P \downarrow P$	$Q \downarrow Q$	$(P \downarrow P) \downarrow (Q \downarrow Q)$	$P \wedge Q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	F	F

This result can also be shown using equivalences:

$$\begin{aligned}
(P \downarrow P) \downarrow (Q \downarrow Q) &= [\neg(P \vee P)] \downarrow [\neg(Q \vee Q)] && \text{(definition of } \downarrow) \\
&= (\neg P) \downarrow (\neg Q) && \text{(absorption)} \\
&= \neg[(\neg P) \vee (\neg Q)] && \text{(definition of } \downarrow) \\
&= [\neg\neg P] \wedge [\neg\neg Q] && \text{(de Morgan)} \\
&= P \wedge Q && \text{(double negative)}
\end{aligned}$$

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2. This is an if-and-only-if statement so we must prove both directions or use a chain of equivalences.
   - Solution 1: First, suppose $A \subseteq B$: we must show that $B^c \subseteq A^c$. Notice that $A \subseteq B$ means that if $x \in A$ then $x \in B$. Taking the contrapositive of this statement shows that if $x \notin B$ then $x \notin A$. Equivalently, this says $x \in B^c$ implies $x \in A^c$ which means precisely that $B^c \subseteq A^c$, as desired.
   - Now suppose $B^c \subseteq A^c$: we must show that $A \subseteq B$. The hypothesis says that $x \in B^c$ implies $x \in A^c$, which equivalently says if $x \notin B$ then $x \notin A$. Taking the contrapositive yields the statement that if $x \in A$ then $x \in B$, which implies $A \subseteq B$ as desired.
   - Solution 2: We use a chain of equivalences: $A \subseteq B \iff x \in A$ implies $x \in B \iff x \notin B$ implies $x \notin A \iff x \in B^c$ implies $x \in A^c \iff B^c \subseteq A^c$.

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3. Each part was worth 3 points.
   (a) We want a prime number n such that $n + 4$ is not prime. Some examples of such n are $n = 2$, 5, 11, 23.
   (b) We want a set A such that $A \subseteq \{1, 2, 3, 4, 5\}$ and $A \cap \{2, 3, 4\}$ is empty. There are several choices: $A = \{1, 5\}$, $A = \{1\}$, $A = \{5\}$, or $A = \emptyset$.
   (c) We want a positive integer n that is not prime but doesn't have two different prime factors. Some examples of such n are $n = 1$ (no prime factors) or n equal to a prime power such as 4, 8, 9, 16, or 25.
   (d) In words this statement says "for all real x there exists a real y such that $xy = 1$". For most values of x there is such a value of y, namely, $y = 1/x$. But for $x = 0$, there exists no y with $xy = 1$ because xy is always 0 no matter what y is. Thus, $x = 0$ is a counterexample (in fact, the only one).

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4. We use (strong) induction on n.
   - For the base cases we take $n = 0$ and $n = 1$: we have $a_0 = 4 = 2^0 + 3$ and $a_1 = 5 = 2^1 + 3$ as required.
   - For the inductive step, suppose that $a_{n-1} = 2^{n-1} + 3$ and $a_{n-2} = 2^{n-2} + 3$. Then by definition we see $a_n = 3a_{n-1} - 2a_{n-2} = 3(2^{n-1} + 3) - 2(2^{n-2} + 3) = 3 \cdot 2^{n-1} + 9 - 2 \cdot 2^{n-2} - 6 = 2 \cdot 2^{n-1} + 3 = 2^n + 3$, as required.
   - Therefore, the result holds for all $n \geq 0$ by induction.
   - Remark: Common errors included starting with the wrong base case, failing to do 2 base cases, and writing the steps in the proof of the inductive step in the wrong order.

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5. Part (a) was worth 5 points while (b) and (c) were worth 4 points.

   (a) We use the Euclidean algorithm:

   $$\begin{aligned} 230 &= 10 \cdot 22 + 10 \\ 22 &= 2 \cdot 10 + 2 \\ 10 &= 5 \cdot 2 \end{aligned}$$

   The last nonzero remainder is 2, so the greatest common divisor is $\boxed{2}$.

   (b) We have $\text{lcm}(a, b) = ab/\gcd(a, b) = \boxed{460 \cdot 44/4 = 5060}$.

   (c) We use the extended Euclidean algorithm to solve for the remainders one at a time. This yields

   $$\begin{aligned} 10 &= 230 - 10 \cdot 22 \\ 2 &= 22 - 2 \cdot 10 = 22 - 2 \cdot (230 - 10 \cdot 22) = -2 \cdot 230 + 21 \cdot 22 \end{aligned}$$

   Therefore, we can take $x = \boxed{-2}$ and $y = \boxed{21}$.

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6. This is an if-and-only-if statement so we must prove both directions or use a chain of equivalences.

   - Solution 1: First suppose that $\gcd(a, b) = a$. Then since the gcd of two numbers is a divisor of both, that means $a|a$ and $a|b$, so in particular $a|b$.
   - Conversely, suppose $a|b$. Then a is a common divisor of a and b, and also any other common divisor d must divide a hence $d \le a$: therefore, a is the greatest common divisor of a and b.
   - Solution 2: If $\gcd(a, b) = a$ then as in Solution 1 that means $a|b$. Conversely, if $a|b$ then $b = ka$ for some integer k, and then $\gcd(a, b) = \gcd(a, ka) = a \cdot \gcd(1, k) = a$ by gcd properties.
   - Solution 3: Suppose a and b have prime factorizations $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $b = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$. Then $\gcd(a, b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \cdots p_k^{\min(a_k,b_k)}$ which equals a if and only if $a_i = \min(a_i, b_i)$ for each i. But that is true if and only if $a_i \le b_i$, which is in turn true if and only if $a|b$.

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7. Parts (a)-(c) were each worth 3 points, while each item in (d) was worth 1 point.

   (a) Symbolically, the given statement is $\forall x \in A\, \exists y \in B,\ (x + y) \in C$. Negating flips each quantifier to the other type (but preserves their order), so the negation is $\boxed{\text{(i) } \exists x \in A\, \forall y \in B,\ (x + y) \notin C}$.

   (b) The mistake is that $\boxed{\text{(ii) The proof only establishes one implication of the biconditional.}}$ The statement of the proposition is an if-and-only-if statement, but only the implication "if m is odd then m^2 is odd" is shown (the converse is not). The other three choices do not apply: it is valid to start showing the conditional by assuming m is odd, there is no justification needed for why $m = 2k+1$ (it is the definition of an odd number), and the proof does not need to start by assuming m^2 is odd (that would be for a proof of the converse).

   (c) The mistake is that $\boxed{\text{(i) The argument starts with the wrong base case.}}$ The proposition claims the result for all positive integers n, but the base case starts with $n = 2$ instead of $n = 1$. The other three choices do not apply: the argument does not need two base cases, the result in the inductive step is correctly shown as $1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = (n + 1)^2$, and the inductive step should not start by assuming that statement.

   (d) In order, the responses are

      i. $\boxed{\text{True}}$, since $(\neg F \wedge T) \Rightarrow (\neg F)$ resolves to $F \Rightarrow T$ which is true.

      ii. $\boxed{\text{False}}$ because the set $\{2\}$ is not an element of A (the number 2 is, but $2 \neq \{2\}$).

      iii. $\boxed{\text{True}}$ because the empty set is a subset of every set, including A.

      iv. $\boxed{\text{True}}$ by the formula for the gcd in terms of a prime factorization, the exponent of each prime in the gcd is the minimum of the corresponding exponents in the given integers.

      v. $\boxed{\text{False}}$ because having $2^a = 5^b$ for positive integers a, b would contradict the uniqueness of prime factorizations.

      vi. $\boxed{\text{True}}$, as proven by Euclid and shown in class, there are infinitely many prime numbers.
