Math 1465: Intensive Mathematical Reasoning

Midterm 1 (Instructor: Dummit) – Form A

October 16th, 2024

NAME (please print legibly): ______________________________

- **Show all work and justify all answers. A correct answer without sufficient work may not receive full credit!**
- **You may appeal to results covered at any point in the course, but please make clear what results you are using. [Box] all final numerical answers.**
- **In problems with multiple parts, you may use the results of previous parts in later parts, even if you did not solve the earlier parts correctly.**
- **You are responsible for checking that this exam has all 8 pages.**
- **You are allowed a 1-page note sheet and a calculator. Time limit: [65 minutes].**

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	10	
2	10	
3	12	
4	10	
5	13	
6	10	
7	15	
TOTAL	80	

1. (10 points) Suppose the logical operator $\downarrow$ is defined so that $P \downarrow Q = \neg(P \vee Q)$. Show that $(P \downarrow P) \downarrow (Q \downarrow Q)$ is logically equivalent to $P \wedge Q$.

Note: All mathematically valid solution methods are acceptable.

2. (10 points) Suppose that A and B are arbitrary subsets of a universal set U, and recall that the complement $A^c = \{x \in U : x \notin A\}$ consists of the elements not in A.

Prove that $A \subseteq B$ if and only if $B^c \subseteq A^c$.

Note: You may use a Venn diagram or an example to illustrate your proof, but simply appealing to a Venn diagram or an example will NOT receive credit.

3. (12 points) Find a counterexample to each of the following statements (make sure to include brief justification for why your counterexample is valid):

(a) For all prime numbers n, the number $n + 4$ is also prime.

(b) There exists no set A such that $A \subseteq \{1, 2, 3, 4, 5\}$ and $A \cap \{2, 3, 4\} = \emptyset$.

(c) If n is a positive integer that is not prime, then n has at least two different prime factors.

(d) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1$.

4. (10 points) The sequence $a_0, a_1, a_2, a_3, \ldots$ is defined by setting $a_0 = 4$, $a_1 = 5$, $a_2 = 7$, and for $n \geq 2$, $a_n = 3a_{n-1} - 2a_{n-2}$. Prove that $a_n = 2^n + 3$ for all integers $n \geq 0$.

5. (13 points) Let $a = 230$ and $b = 22$.

(a) Find the greatest common divisor $\gcd(a, b)$.

(b) Find the least common multiple $\operatorname{lcm}(a, b)$. (You do not need to simplify your answer.)

(c) Find integers x and y such that $xa + yb = \gcd(a, b)$.

6. (10 points) Suppose a, b are positive integers. Prove that $\gcd(a, b) = a$ if and only if $a|b$.

Note: You may use an example to illustrate your proof, but simply appealing to an example will NOT receive credit.

7. (15 points) Circle the correct response for each of the following (no work is required, and there is no partial credit or penalty for incorrect answers):

(a) Which of the following is the negation of the statement "For all $x \in A$, there exists a $y \in B$ such that $x + y \in C$"?

(i) $\exists x \in A \, \forall y \in B, (x + y) \notin C$.

(ii) $\forall y \in B \, \exists x \in A, (x + y) \notin C$.

(iii) $\forall x \notin A \, \exists y \notin B, (x + y) \notin C$.

(iv) $\exists x \notin B \, \exists y \notin A, (x + y) \in C$.

(b) What is the mistake in the following claimed proof?

Proposition: An integer m is odd if and only if m^2 is odd.

Proof: Suppose m is odd, so that $m = 2k + 1$ for an integer k. Then we can see that $m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is also odd.

(i) The proof assumes m is odd without justification.

(ii) The proof only establishes one implication of the biconditional.

(iii) The proof should start by assuming m^2 is odd instead of getting it as the conclusion.

(iv) The proof does not justify why $m = 2k + 1$.

(c) What is the mistake in the following claimed proof?

Proposition: For all positive integers n, we have $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Proof: Induct on n. The base case $n = 2$ holds since $1 + 3 = 2^2$.

To show the inductive step, suppose that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$: then we have $1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2$, as required.

(i) The argument starts with the wrong base case.

(ii) The inductive step should show $1 + 3 + 5 + \cdots + (2n - 1) + 2n = (n + 1)^2$ instead.

(iii) The inductive step should start by assuming that $1+3+5+\cdots+(2n+1) = (n+1)^2$.

(iv) The argument should use two base cases.

(d) Identify each statement as true or false (each answer is worth independent points):

True	**False**	If P and Q are both false, then $(\neg P \wedge Q) \Rightarrow (\neg Q)$ is true.
True	**False**	If $A = \{1, \{\emptyset\}, 2\}$ then $\{2\} \in A$.
True	**False**	If $A = \{1, \{\emptyset\}, 2\}$ then $\emptyset \subseteq A$.
True	**False**	The greatest common divisor of $2^3 \cdot 3^5 \cdot 5^4$ and $2^4 \cdot 3^3 \cdot 7$ is $2^3 \cdot 3^3$.
True	**False**	There exist positive integers a and b with $2^a = 5^b$.
True	**False**	There are infinitely many prime numbers.
