1. Suppose the logical operator $*$ is defined so that $P * Q = \neg P \wedge Q$. Using a truth table or otherwise, determine whether the following pairs of statements are logically equivalent for arbitrary propositions P, Q, and R:
   (a) $P * (P * P)$ and $\neg P$.
   (b) $(P \Rightarrow Q) \Rightarrow (P * Q)$ and True.
   (c) $P * Q$ and $\neg(Q \Rightarrow P)$.
   (d) $(Q * Q) * P$ and $P \wedge Q$.
   (e) $(P * Q) * R$ and $P * (Q * R)$.
   (f) $(P * Q) \vee R$ and $(R * \neg Q) * (P * R)$.

2. Suppose that $A = \{1, 2, \{1\}, \{2\}, \{1, 3\}\}$ and $B = \{1, \{1, 2\}, \{1, 3\}, \{2\}\}$. Find the truth value of each statement:
   (a) $1 \in A$.
   (b) $1 \subseteq A$.
   (c) $\{1\} \in B$.
   (d) $\{1\} \subseteq B$.
   (e) $\{1\} \in A \cap B$.
   (f) $\{1\} \subseteq A \cap B$.
   (g) $\{1, 2\} \in A \cup B$.
   (h) $\{1, 2\} \subseteq A \cup B$.
   (i) $\{1, 3\} \in A \cap B$.
   (j) $\{1, 3\} \subseteq A \cap B$.

3. Suppose that $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 4, 8\}$, and $C = \{2, 3, 5, 7\}$ with universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find:
   (a) $A \cap B$.
   (b) $A \cup C$.
   (c) $A^c \cap C^c$.
   (d) $B^c \cup (A \cap C)$.
   (e) $(B \cap C) \cap (A \cup B^c)$.
   (f) $A \times (B \cap C)$.
   (g) $(A \cap B) \times (B \cap C)$.
   (h) $(A \times C) \cap (C \times A)$.

4. Write a negation for each of the following statements:
   (a) $\forall x \forall y \exists z,\ x + y + z > 5$.
   (b) Every integer is a rational number.
   (c) $\forall x \in A\, \forall y \in B,\ x \cdot y \in A \cap B$.
   (d) There is a perfect square that is not even.
   (e) The integer n is a prime number and $n < 10$.
   (f) $\forall \epsilon > 0\, \exists \delta > 0,\ (|x - a| < \delta) \Rightarrow (|x^2 - a^2| < \epsilon)$.
   (g) For any $x \in \mathbb{R}$ there exists an $n \in \mathbb{Z}$ such that $x < n$.
   (h) There exist positive integers a and b with $\sqrt[3]{2} = a/b$.

5. Find the truth values of the following statements, where the universal set is $\mathbb{R}$:
   (a) $\forall x \forall y,\ y \neq x$.
   (b) $\forall x \exists y,\ y \neq x$.
   (c) $\exists x \forall y,\ y \neq x$.
   (d) $\exists x \exists y,\ y \neq x$.
   (e) $\forall x \forall y,\ y^2 \geq x$.
   (f) $\forall x \exists y,\ y^2 \geq x$.
   (g) $\exists x \forall y,\ y^2 \geq x$.
   (h) $\exists x \exists y,\ y^2 \geq x$.

6. Calculate the following things:
   (a) The gcd and lcm of 256 and 520.
   (b) The gcd and lcm of 921 and 177.
   (c) The gcd and lcm of 2019 and 5678.
   (d) The gcd and lcm of $2^3 3^2 5^4 7$ and $2^4 3^3 5^4 11$.

7. Suppose A, B, and C are arbitrary sets contained in a universal set U. Identify which statements are true and which are false. Then prove the true statements and give a counterexample for the false ones.
   (a) $(A \cup B) \backslash A = B \backslash A$.
   (b) $A \backslash (B \cap C) = (A \backslash B) \cap (A \backslash C)$.
   (c) $(A \cap B)^c \cup B \subseteq (A^c \cup B)^c$.
   (d) $A^c \cap B^c \subseteq (A \backslash B)^c \cap (B \backslash A)^c$.

8. Write, and then prove, the contrapositive of each of these statements (assume n refers to an integer):
   (a) If a and b are integers, then $3a - 9b \neq 2$.
   (b) Suppose $a, b \in \mathbb{Z}$. If $ab = 1$ then $a \leq 1$ or $b \leq 1$.
   (c) If $5n + 1$ is even, then n is odd.
   (d) If n^3 is odd, then n is odd.
   (e) If n is not a multiple of 3, then n cannot be written as the sum of 3 consecutive integers.
   (f) Suppose $a, b \in \mathbb{Z}$. If n does not divide ab, then n does not divide a and n does not divide b.

9. Find a counterexample to each of the following statements:

(a) For any integers a, b, and c, if $a|b$ and $a|c$, then $b|c$.

(b) If p and q are prime, then $p+q$ is never prime.

(c) If n is an integer, then n^2+n+11 is always prime.

(d) There do not exist integers a and b with $a^2-b^2=7$.

(e) The sum of two irrational numbers is always irrational.

(f) If $n>1$ is an integer, then $\sqrt{n}$ is always irrational.

(g) If $n \neq 3$ then $n^2 \neq 9$.

(h) There are no positive integers m,n with $m^2-2n^2=1$.

(i) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $y^4=x$.

(j) The sum of two perfect squares is never a perfect cube.

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10. Prove the following (recall the Fibonacci numbers F_i are defined by $F_1=F_2=1$ and $F_{n+1}=F_n+F_{n-1}$ for all $n \geq 2$):

(a) If F_n is the nth Fibonacci number, prove that $F_1+F_3+F_5+\cdots+F_{2n+1}=F_{2n+2}$ for every positive integer n.

(b) Suppose n is an integer. Prove that $2|n$ and $3|n$ if and only if $6|n$.

(c) Prove that $1+\dfrac{1}{2}+\dfrac{1}{4}+\cdots+\dfrac{1}{2^n}=2-\dfrac{1}{2^n}$ for every positive integer n.

(d) For any sets A,B,C, prove $A\backslash(B\cap C)=(A\backslash B)\cup(A\backslash C)$.

(e) Suppose p is a prime and a is a positive integer. If $p|a^2$, prove that $p|a$.

(f) Prove there do not exist integers a and b such that $a^2=33+9b$. [Hint: Use (e).]

(g) Prove that any two consecutive perfect squares (i.e., k^2 and $(k+1)^2$) are relatively prime. [Hint: Use (e).]

(h) Prove that $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n\cdot(n+1)}=\dfrac{n}{n+1}$ for every positive integer n.

(i) If $A=\{4a+6b : a,b \in \mathbb{Z}\}$ and $B=\{2c : c \in \mathbb{Z}\}$, prove that $A=B$.

(j) Show that the propositions $\neg[Q \wedge \neg(P \wedge Q)] \wedge \neg P$ and $\neg Q \wedge \neg P$ are logically equivalent.

(k) If p is a prime, prove that $\gcd(n,n+p)>1$ if and only if $p|n$.

(l) Suppose $a_1=1$ and $a_n=3a_{n-1}+4$ for all $n \geq 2$. Prove that $a_n=3^n-2$ for every positive integer n.

(m) If $C=\{6c : c \in \mathbb{Z}\}$ and $D=\{10a+14b : a,b \in \mathbb{Z}\}$, prove that $C \subseteq D$.

(n) Suppose $b_1=3$ and $b_{n+1}=2b_n-n+1$ for all $n \geq 2$. Prove that $b_n=2^n+n$ for every positive integer n.

(o) Suppose $c_1=c_2=2$, and for all $n \geq 3$, $c_n=c_{n-1}c_{n-2}$. Prove that $c_n=2^{F_n}$ for every positive integer n.

(p) Suppose $d_1=2$, $d_2=4$, and for all $n \geq 3$, $d_n=d_{n-1}+2d_{n-2}$. Prove that $d_n=2^n$ for every positive integer n.

(q) For any sets A,B inside a universal set U, prove $A \cup B^c=U$ if and only if $A^c \cap B=\emptyset$.

(r) Show that 7^n-1 is a multiple of 6 for every positive integer n.

(s) Prove that a and b are relatively prime if and only if a^2 and b^2 are relatively prime. [Hint: Use (e).]

(t) For any sets A,B,C, prove $A \subseteq B \cup C$ if and only if $A\backslash B \subseteq C$.
