1. (a) Not equivalent (b) Not equivalent (c) Equivalent (d) Not equivalent (e) Not equivalent (f) Equivalent

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2. (a) True (b) False (c) False (d) True (e) False (f) True (g) True (h) True (i) True (j) False

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3. (a) $\{1\}$ (b) $\{1, 2, 3, 5, 7, 9\}$ (c) $\{4, 6, 8\}$ (d) $\{3, 5, 6, 7, 9\}$ (e) $\emptyset = \{\}$ (f) $\{(1, 2), (3, 2), (5, 2), (7, 2), (9, 2)\}$
(g) $\{(1, 2)\}$ (h) $\{(3, 3), (3, 5), (3, 7), (5, 3), (5, 5), (5, 7), (7, 3), (7, 5), (7, 7)\}$

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4. (a) $\exists x \exists y \forall z,\ x + y + z \leq 5$
(b) There exists an integer that is not a rational number.
(c) $\exists x \in A\, \exists y \in B,\ x \cdot y \notin A \cap B$.
(d) Every perfect square is even.
(e) The integer n is either not prime or $n \geq 10$.
(f) $\exists \epsilon > 0\, \forall \delta > 0,\ (|x - a| < \delta) \wedge (|x^2 - a^2| \geq \epsilon)$.
(g) There exists an $x \in \mathbb{R}$ such that for all $n \in \mathbb{Z}$, $x \geq n$.
(h) For all positive integers a and b, $\sqrt[3]{2} \neq a/b$.

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5. (a) False (b) True (c) False (d) True (e) False (f) True (g) True (h) True

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6. (a) gcd 8, lcm $256 \cdot 520/8$. (b) gcd 3, lcm $921 \cdot 177/3$. (c) gcd 1, lcm $2019 \cdot 5678$. (d) gcd $2^3 3^2 5^4$, lcm $2^4 3^3 5^4 7 \cdot 11$.

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7. (a) True. Note $x \in (A \cup B)\backslash A$ iff $x \in (A \cup B) \cap A^c$ iff $x \in B \cap A^c$ iff $x \in B \backslash A$.
(b) False. Counterexample: $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$. Then $A\backslash(B \cap C) = \{1, 2\}$ while $(A\backslash B) \cap (A\backslash C) = \emptyset$.
(d) False. Counterexample: $A = \{1\}$, $B = \{1, 2\}$ with $U = \{1, 2\}$. Then $(A \cap B)^c \cup B = \{1, 2\}$ while $(A^c \cap B)^c = \{1\}$.
(e) True. Note $(A\backslash B)^c = (A \cap B^c)^c = A^c \cup B$, and similarly $(B\backslash A)^c = A \cup B^c$. If $x \in A^c \cap B^c$ then $x \in A^c \cup B$ and also $x \in A \cup B^c$.

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8. (a) If $3a - 9b = 2$, then a and b cannot both be integers. Proof: By contradiction, if a and b are integers, then 3 divides $3a - 9b$ but 3 does not divide 2 (impossible).
(b) If $a > 1$ and $b > 1$, then $ab \neq 1$. Proof: If $a > 1$ and $b > 1$ then multiplying $a > b$ by b yields $ab > b > 1$ so $ab > 1$. In particular $ab \neq 1$.
(c) If n is even, then $5n + 1$ is odd. Proof: If $n = 2k$ then $5n + 1 = 10k + 1 = 2(5k) + 1$ is odd by definition.
(d) If n is even then n^3 is even. Proof: If $n = 2k$ then $n^3 = 8k^3 = 2(4k^3)$ is even by definition.
(e) If n is the sum of 3 consecutive integers, then n is a multiple of 3. Proof: If $n = a + (a + 1) + (a + 2)$ then $n = 3a + 3 = 3(a + 1)$ is a multiple of 3.
(f) If n divides a or n divides b then n divides ab. Proof: If $n|a$ then $a = kn$ so $ab = (kb)n$, and if $n|b$ then $b = ln$ so $ab = (al)n$. In either case, $n|ab$.

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9. There are many examples for each part. Here is one for each:
(a) Example: $a = 2$, $b = 4$, $c = 6$.
(b) Example: $p = 2$, $q = 3$, then $p + q = 5$ is prime.
(c) Example: $n = 11$, then $n^2 + n + 11 = 11 \cdot 13$ is not prime.
(d) Example: $a = 4$, $b = 3$, then $a^2 - b^2 = 16 - 9 = 7$.
(e) Example: $\sqrt{2} + (-\sqrt{2}) = 0$ is rational, but $\sqrt{2}$ and $-\sqrt{2}$ are irrational.
(f) Example: $\sqrt{4} = 2$ is rational.
(g) Example: $n = -3$, then $n \neq 3$ but $n^2 = 9$.
(h) Example: $m = 3$, $n = 2$, then $m^2 - 2n^2 = 9 - 8 = 1$.
(i) Example: $x = -1$, then there is no possible y with $y^4 = x$.
(j) Examples: $2^2 + 2^2 = 2^3$, or $5^2 + 10^2 = 5^3$.

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10. Here are brief outlines of each proof:

(a) Induct on n. Base case $n = 1$ has $F_1 + F_3 = 3 = F_4$. Inductive step: if $F_1 + \cdots + F_{2n+1} = F_{2n+2}$ then $F_1 + \cdots + F_{2n+1} + F_{2n+3} = [F_1 + \cdots + F_{2n+1}] + F_{2n+3} = F_{2n+2} + F_{2n+3} = F_{2n+4}$ as required.

(b) Clearly, if $6|n$ then $2|n$ and $3|n$. For the other direction, if $2|n$ then $n = 2k$. Then if $3|2k$ we must have $3|k$ since $3 \nmid 2$ and 3 is prime. So $k = 3a$, and thus $n = 6a$, meaning $6|n$.

(c) Induct on n. Base case $n = 1$ has $1 = 2 - 1/2^0$. Inductive step: If $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$, then $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^{n+1}}$ as required.

(d) Note $x \in A\backslash(B \cap C) \iff x \in A$ and $x \notin (B \cap C) \iff x \in A$ and ($x \notin B$ or $x \notin C$) $\iff$ ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$) $\iff x \in A\backslash B$ or $x \in A\backslash C \iff x \in (A\backslash B) \cup (A\backslash C)$.

(e) If $p|a \cdot a$ then $p|a$ or $p|a$ by the prime divisibility property. Since the two conclusion statements are the same, we have $p|a$.

(f) Note that $33 + 9b$ is divisible by 3 but not 9. But then a^2 is divisible by 3 by (d), which would mean $3|a$ and thus $9|a^2$, but this is impossible.

(g) If p is a prime with $p|k^2$ and $p|(k+1)^2$, then by (d) we have $p|k$ and $p|(k+1)$ so that $p|(k+1) - k = 1$, impossible.

(h) Induct on n. Base case $n = 1$ has $\frac{1}{1 \cdot 2} = \frac{1}{2}$. Inductive step: if $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$ then $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$ as required.

(i) First, $A \subseteq B$ because if $n = 4a + 6b$ then $n = 2(2a + 3c) \in B$. Also, $B \subseteq A$ because if $n = 2c$ then we would have $n = 4(2c) + 6(-c) \in A$ via Euclidean algorithm calculation.

(j) $\neg[Q \wedge \neg(P \wedge Q)] \wedge \neg P = [\neg Q \vee (P \wedge Q)] \wedge \neg P = (\neg Q \wedge \neg P) \vee (P \wedge Q \wedge \neg P) = (\neg Q \wedge \neg P) \vee \text{False} = \neg Q \wedge \neg P$. Alternatively, draw a truth table.

(k) Note $\gcd(n, n+p) = \gcd(n, p)$ by gcd properties. Then $\gcd(n, p)$ divides p so is either 1 or p, and it is equal to p if and only if $p|n$ (by definition of gcd).

(l) Induct on n. Base case $n = 1$ has $a_1 = 3^1 - 2$. Inductive step: if $a_n = 3^n - 2$ then $a_{n+1} = 3(3^n - 2) + 4 = 3^{n+1} - 2$ as claimed.

(m) If $n \in C$, then $n = 6c$ for some c. Then $n = 10(2c) + 14(-c) \in D$ as required.

(n) Induct on n. Base case $n = 1$ has $b_1 = 2^1 + 1$. Inductive step: if $b_n = 2^n + n$ then $b_{n+1} = 2(2^n + n) - n + 1 = 2^{n+1} + (n+1)$ as claimed.

(o) Induct on n. Base cases $n = 1$ and $n = 2$ have $c_1 = 2^{F_1}$ and $c_2 = 2^{F_2}$. Inductive step: if $c_n = 2^{F_n}$ and $c_{n-1} = 2^{F_{n-1}}$ then $c_{n+1} = c_n c_{n-1} = 2^{F_n} 2^{F_{n-1}} = 2^{F_n + F_{n-1}} = 2^{F_{n+1}}$ as required.

(p) Induct on n . Base cases $n = 1$ and $n = 2$ have $d_1 = 2^1$ and $d_2 = 2^2$. Inductive step: if $d_n = 2^n$ and $d_{n-1} = 2^{n-1}$ then $d_{n+1} = 2^n + 2(2^{n-1}) = 2^n + 2^n = 2^{n+1}$ as required.

(q) Observe $(A \cup B^c)^c = A^c \cap (B^c)^c = A^c \cap B$ by de Morgan's laws, so $A \cup B^c$ and $A^c \cap B$ are complements. Thus, if $A \cup B^c = U$ then $A^c \cap B = U^c = \emptyset$ and conversely if $A^c \cap B = \emptyset$ then $A \cup B^c = \emptyset^c = U$.

(r) Induct on n. Base case $n = 1$ has $7^1 - 1 = 6$ a multiple of 6. Inductive step: if 6 divides $7^n - 1$ then 6 divides $7(7^n - 1) + 6 = 7^{n+1} - 1$.

(s) Show the contrapositive. If a, b are not relatively prime so that $d|a$ and $d|b$ for some $d > 1$, then $d^2|a^2$ and $d^2|b^2$ so a^2, b^2 are not relatively prime. Conversely by (e) if p is prime and $p|a^2$ and $p|b^2$ then $p|a$ and $p|b$ so a, b are not relatively prime.

(t) First suppose $A \subseteq B \cup C$. If $x \in A\backslash B$ then $x \in A$ and $x \notin B$. Since $A \subseteq B \cup C$, $x \in B \cup C$ so $x \in B$ or $x \in C$ but since $x \notin B$ we must have $x \in C$: thus $A\backslash B \subseteq C$. Conversely suppose $A\backslash B \subseteq C$ and let $x \in A$. If $x \in B$ then clearly $x \in B \cup C$ and otherwise if $x \notin B$ then $x \in A\backslash B$ hence $x \in C$ and once again $x \in B \cup C$: thus $A \subseteq B \cup C$.