1. For each partial ordering on each set, decide whether or not the relation is a total ordering, and briefly explain your reasoning.

   (a) The alphabetical-order relation $\{(a,a),(a,b),(a,c),(b,b),(b,c),(c,c)\}$ on the set $\{a,b,c\}$.

   - This $\boxed{\text{is a total ordering}}$: any two elements a,b,c are comparable.

   (b) The identity relation $\{(1,1),(2,2),(3,3),(4,4)\}$ on the set $\{1,2,3,4\}$.

   - This $\boxed{\text{is not a total ordering}}$: for example 1 and 2 are not comparable (indeed any element is only comparable to itself).

   (c) The divisibility relation on the set $\{1,2,3,4,5,\dots\}$ of positive integers.

   - This $\boxed{\text{is not a total ordering}}$: for example 2 and 3 are not comparable since neither divides the other.

   (d) The divisibility relation on the set $\{1,10,100,1000,\dots\}$ of powers of 10.

   - This $\boxed{\text{is a total ordering}}$: for any two powers of 10, the smaller will divide the larger.

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2. For each f, A, and B, identify whether or not f is a function from A to B.

   (a) $A = \{1,2,3\}$, $B = \{4\}$, where $f = \{(1,4),(2,4),(3,4)\}$.

   - This $\boxed{\text{is a function from } A \text{ to } B}$: each element of A is the first coordinate of exactly one pair, and all second coordinates are in B.

   (b) $A = \{1\}$, $B = \{2,3,4\}$, where $f = \{(1,2),(1,3),(1,4)\}$.

   - This $\boxed{\text{is not a function from } A \text{ to } B}$, because it is not well-defined on 1 (it attempts to map 1 to three different values).

   (c) $A = \{1,2,3\}$, $B = \{4\}$, where $f = \{(1,2),(2,3),(3,4)\}$.

   - This $\boxed{\text{is not a function from } A \text{ to } B}$, because it maps 1 and 2 to values that are not in B.

   (d) $A = \{1,2,3\}$, $B = \{2,3,4\}$, where $f = \{(1,2),(2,3),(3,4)\}$.

   - This $\boxed{\text{is a function from } A \text{ to } B}$: each element of A is the first coordinate of exactly one pair, and all second coordinates are in B.

   (e) $A = \{1,2,3\}$, $B = \{2,3,4,5,6\}$, where $f = \{(1,2),(2,3),(3,4)\}$.

   - This $\boxed{\text{is a function from } A \text{ to } B}$: each element of A is the first coordinate of exactly one pair, and all second coordinates are in B.

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3. For each f, A, and B, identify whether or not f is a well-defined function from A to B. (Hint: Exactly three of them are well defined.)

   (a) $A = \mathbb{Q}$, $B = \mathbb{Q}$, where $f(a/b) = a/b^2$.

   - This $\boxed{\text{is not a function from } A \text{ to } B}$, because the definition is ambiguous.
   - For example, the definition says $f(1/2) = 1/4$ while $f(2/4) = 2/16 = 1/8$, but $1/2 = 2/4$ as rational numbers. So it does not give a clear value for f on the input value $1/2$.

   (b) $A = \mathbb{Q}$, $B = \mathbb{Q}$, where $f(a/b) = a^2/b^2$.

   - This $\boxed{\text{is a function from } A \text{ to } B}$, because it is another name for the squaring function $f(x) = x^2$.
   - Here, if we use different representations of a rational number, the output is the same. For example, the definition says $f(1/2) = 1/4$ while $f(2/4) = 4/16 = 1/4$, which is the same.

(c) $A = \mathbb{Z}$, $B = \mathbb{Z}/m\mathbb{Z}$, where $f(a) = \overline{a}$, with $m > 1$ a fixed modulus.

- This $\boxed{\text{is a function from } A \text{ to } B}$, because for each integer $a \in \mathbb{Z}$ there is a unique residue class $\overline{a}$ modulo m that it lies in, so the function is well-defined.

(d) $A = \mathbb{Z}/m\mathbb{Z}$, $B = \mathbb{Z}$, where $f(\overline{a}) = a$, with $m > 1$ a fixed modulus.

- This $\boxed{\text{is not a function from } A \text{ to } B}$, because it is not well-defined. Explicitly, note that $\overline{0} = \overline{m}$ as residue classes, but the definition would say $f(\overline{0}) = 0$ while $f(\overline{m}) = m$, so f does not assign a well-defined value to the residue class $\overline{0}$.

(e) $A = \mathbb{Z}/m\mathbb{Z}$, $B = \mathbb{Z}/m\mathbb{Z}$, where $f(\overline{a}) = \overline{a^2}$, with $m > 1$ a fixed modulus.

- This $\boxed{\text{is a function from } A \text{ to } B}$, because it is another name for the squaring function $f(x) = x^2$.
- Explicitly, for each residue class $\overline{a}$, the residue class $\overline{a^2} = \overline{a}^2$ is well defined, because multiplication of residue classes is well defined.

(f) $A = \mathbb{Z}/m\mathbb{Z}$, $B = \mathbb{Z}/m\mathbb{Z}$, where $f(\overline{a}) = \overline{a}^{-1}$, with $m > 1$ a fixed modulus.

- This $\boxed{\text{is not a function from } A \text{ to } B}$, because it is only defined when the residue class $\overline{a}$ has a multiplicative inverse, and this is only true when a is relatively prime to m. For instance, $f(\overline{0})$ is not defined, because $\overline{0}^{-1}$ does not exist.

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4. For each function $f : A \to B$, determine whether f is (i) one-to-one, (ii) onto, and (iii) a bijection.

(a) $f_1 = \{(1, 4), (2, 5), (3, 6)\}$ from $A = \{1, 2, 3\}$ to $B = \{4, 5, 6\}$.

- No element of B is the second coordinate of more than one ordered pair, so f is $\boxed{\text{one-to-one}}$.
- Each element of B is the second coordinate of at least one ordered pair, so f is $\boxed{\text{onto}}$.
- Then since f is both one-to-one and onto, it is a $\boxed{\text{bijection}}$.

(b) $f_2 = \{(1, 4), (2, 5), (3, 6)\}$ from $A = \{1, 2, 3\}$ to $B = \{4, 5, 6, 7\}$.

- No element of B is the second coordinate of more than one ordered pair, so f is $\boxed{\text{one-to-one}}$.
- However, the element $7 \in B$ is not the second coordinate of any ordered pair, so f is $\boxed{\text{not onto}}$ hence $\boxed{\text{not a bijection}}$.

(c) $f_3 = \{(1, 5), (2, 6), (3, 6), (4, 6)\}$ from $A = \{1, 2, 3, 4\}$ to $B = \{5, 6\}$.

- Here we can see that the element $2 \in B$ is the second coordinate of multiple pairs, so f is $\boxed{\text{not one-to-one}}$ hence $\boxed{\text{not a bijection}}$.
- But each element of B is the second coordinate of at least one ordered pair, so f is $\boxed{\text{onto}}$.

(d) $f_4(x) = 2x + 1$ from $A = \mathbb{R}$ to $B = \mathbb{R}$.

- This function is $\boxed{\text{one-to-one}}$ and $\boxed{\text{onto}}$ hence a $\boxed{\text{bijection}}$.
- One may verify both properties separately, or observe that f_4 has an inverse function given by $f^{-1}(x) = \frac{1}{2}(x - 1)$, which shows directly that is a bijection.

(e) $f_5(n) = 2n + 1$ from $A = \mathbb{Z}$ to $B = \mathbb{Z}$.

- This function is $\boxed{\text{one-to-one}}$ because $2n + 1 = 2m + 1$ implies $m = n$ for integers m, n.
- However, it is $\boxed{\text{not onto}}$ hence $\boxed{\text{not a bijection}}$ because its image consists only of the odd integers. For a specific example, there is no integer solution to $2n + 1 = 0$.

(f) $f_6(n) = \dfrac{1}{n^2 + 1}$ from $A = \mathbb{Z}$ to $B = \mathbb{Q}$.

- This function is $\boxed{\text{not one-to-one}}$ hence $\boxed{\text{not a bijection}}$: for example we have $f(1) = 1/2 = f(-1)$.
- It is also is $\boxed{\text{not onto}}$, since for example there is no integer n for which $f(n) = 2$ (for example): solving $f(n) = 2$ yields $n^2 = -1/2$, which does not even have any real solutions.

(g) $f_7(a) = \overline{a}$ from $A = \mathbb{Z}$ to $B = \mathbb{Z}/m\mathbb{Z}$, with $m > 1$ a fixed modulus.

- This function is $\boxed{\text{not one-to-one}}$ hence $\boxed{\text{not a bijection}}$: for example we have $f(0) = \overline{0} = \overline{m} = f(m)$.
- However, it is $\boxed{\text{onto}}$: for any residue class $\overline{a} \in \mathbb{Z}/m\mathbb{Z}$, we clearly have $f(a) = \overline{a}$.

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5. Show the following:

(a) Suppose R is a partial ordering on a set A. Show that R^{-1} is also a partial ordering on A.

- Reflexive: Since R is a relation on A, for any $a \in A$ we have $(a, a) \in R$. Thus $(a, a) \in R^{-1}$ and so R^{-1} is reflexive.
- Antisymmetric: Suppose that $(a, b) \in R^{-1}$ and $(b, a) \in R^{-1}$. By definition $(b, a) \in R$ and $(a, b) \in R$ so since R is antisymmetric we have $a = b$. Thus R^{-1} is also antisymmetric.
- Transitive: Suppose that $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$. By definition $(b, a) \in R$ and $(c, b) \in R$ so since R is transitive we have $(c, a) \in R$ and thus $(a, c) \in R^{-1}$. So R^{-1} is transitive.

(b) Suppose R is a total ordering on a set A. Show that R^{-1} is also a total ordering on A.

- By (a) R^{-1} is a partial ordering so we just need to show any two elements are comparable.
- For any $a, b \in A$ since R is a total ordering either $(a, b) \in R$ or $(b, a) \in R$. But this implies $(b, a) \in R^{-1}$ or $(a, b) \in R^{-1}$, and so a is comparable to b under R^{-1} as well.

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6. Suppose $f : A \to B$ is a function and S is an equivalence relation on B. Prove that the relation $R : A \to A$ given by $R = \{(a, b) \in A \times A : (f(a), f(b)) \in S\}$ is an equivalence relation on A.

- R is reflexive: for any $a \in A$, we have $(f(a), f(a)) \in S$ because S is reflexive. So by definition, $(a, a) \in R$.
- R is symmetric: suppose $(a, b) \in R$, meaning that $(f(a), f(b)) \in S$. Then because S is symmetric, $(f(b), f(a)) \in S$, and so $(b, a) \in R$ as required.
- R is transitive: suppose $(a, b) \in R$ and $(b, c) \in R$. Then $(f(a), f(b)) \in S$ and also $(f(b), f(c)) \in S$. Then because S is transitive, $(f(a), f(c)) \in S$, and so $(a, c) \in R$ as required.
- Hence R is reflexive, symmetric, and transitive, so it is an equivalence relation.

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7. Suppose A, B, and C are sets.

(a) If $f : B \to C$ and $g : A \to B$ are both one-to-one, prove that $f \circ g$ is also one-to-one.

- Suppose that $(f \circ g)(a_1) = (f \circ g)(a_2)$ for some $a_1, a_2 \in A$.
- By the definition of composition, this means $f(g(a_1)) = f(g(a_2))$.
- Since f is one-to-one, $f(g(a_1)) = f(g(a_2))$ implies $g(a_1) = g(a_2)$, and then since g is one-to-one, we have $a_1 = a_2$ as required.

(b) If $f : B \to C$ and $g : A \to B$ are both onto, prove that $f \circ g$ is also onto.

- Let $c \in C$ be arbitrary. Since f is onto, there exists $b \in B$ such that $f(b) = c$.
- Then since g is onto, there exists $a \in A$ such that $g(a) = b$.
- Then we have $f(g(a)) = f(b) = c$: this means there exists $a \in A$ such that $(f \circ g)(a) = f(g(a)) = c$, as required.

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8. Suppose $f : A \to B$ is a function.

- If $S \subseteq A$, we write $f(S) = \{f(s) : s \in S\}$ and call $f(S)$ the image of S.
- If $T \subseteq B$, we write $f^{-1}(T) = \{a \in A : f(a) \in T\}$ and call $f^{-1}(T)$ the inverse image of T.
- When $T = \{b\}$ is a single element, we write $f^{-1}(T)$ as $f^{-1}(b)$ rather than $f^{-1}(\{b\})$, with the understanding that $f^{-1}(b)$ is a set that could be empty or contain more than one element.

Example: For the function $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = x^4$, with $A = \{1, 4\}$ we have $h(A) = \{1, 64\}$ and $h^{-1}(A) = \{-\sqrt{2}, -1, 1, \sqrt{2}\}$. We also have $h(\{-1\}) = h(\{1\}) = \{1\}$ while $h^{-1}(1) = \{1, -1\}$ and $h^{-1}(-1) = \emptyset$.

(a) Suppose $g : \mathbb{R} \to \mathbb{R}$ is the function with $g(x) = x^2$ and recall the notation $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ for a closed interval. Match the following ten image or inverse image sets with their values.

Sets: $g(\{1, 2\})$, $g([-1, 2])$, $g([0, 1])$, $g(\emptyset)$, $g^{-1}(0)$, $g^{-1}(1)$, $g^{-1}(\{1, 4\})$, $g^{-1}(-1)$, $g^{-1}([4, 9])$, $g^{-1}([0, 1])$.

Values: $\emptyset$, $\emptyset$, $\{-1, 1\}$, $[0, 4]$, $\{0\}$, $[-1, 1]$, $\{-2, -1, 1, 2\}$, $[0, 1]$, $[-3, -2] \cup [2, 3]$, $\{1, 4\}$.

- $g(\{1, 2\}) = \{1, 4\}$, $g([-1, 2]) = [0, 4]$, $g([0, 1]) = [0, 1]$, $g(\emptyset) = \emptyset$, $g^{-1}(0) = \{0\}$, $g^{-1}(1) = \{-1, 1\}$, $g^{-1}(\{1, 4\}) = \{-2, -1, 1, 2\}$, $g^{-1}(-1) = \emptyset$, $g^{-1}([4, 9]) = [-3, -2] \cup [2, 3]$, and $g^{-1}([0, 1]) = [-1, 1]$.

(b) Suppose $f : A \to B$ is a function. If S is any subset of A, show that $S \subseteq f^{-1}(f(S))$.

- Suppose $a \in S$. Then $f(a) \in f(S)$, so by definition we have $a \in f^{-1}(f(S))$.

(c) Find an example of a subset S of $\mathbb{R}$ such that $S \neq g^{-1}(g(S))$ for the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$.

- Various options. One is to take $S = \{1\}$, then $g(S) = \{1\}$ so that $g^{-1}(g(S)) = \{-1, 1\} \neq S$.

(d) Suppose $f : A \to B$ is a function. If T is any subset of B, show that $f(f^{-1}(T)) \subseteq T$.

- Suppose $a \in f^{-1}(T)$. Then $f(a) \in T$ by definition. This holds for all $a \in f^{-1}(T)$, so $f(f^{-1}(T)) \subseteq T$.

(e) Find an example of a subset T of $\mathbb{R}$ such that $T \neq g(g^{-1}(T))$ for the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$.

- Various options. One is to take $T = \{-1\}$, then $g^{-1}(T) = \emptyset$ so $g(g^{-1}(T)) = \emptyset$.

(f) Suppose $f : A \to B$ is a function. If B_1 and B_2 are subsets of B, show $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

- Observe that $a \in f^{-1}(B_1 \cap B_2)$ if and only if $f(a) \in B_1 \cap B_2$ if and only if $f(a) \in B_1$ and $f(a) \in B_2$ if and only if $a \in f^{-1}(B_1)$ and $a \in f^{-1}(B_2)$ if and only if $a \in f^{-1}(B_1) \cap f^{-1}(B_2)$.
- Hence the conditions $a \in f^{-1}(B_1 \cap B_2)$ and $a \in f^{-1}(B_1) \cap f^{-1}(B_2)$ are equivalent, so $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(g) Find an example of subsets A_1 and A_2 of $\mathbb{R}$ such that $g(A_1 \cap A_2) \neq g(A_1) \cap g(A_2)$ for the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$.

- Various options. One is to take $A_1 = \{-1\}$ and $A_2 = \{1\}$, then $g(A_1 \cap A_2) = f(\emptyset) = \emptyset$, whereas $g(A_1) = \{1\} = g(A_2)$ so $g(A_1) \cap g(A_2) = \{1\}$.
