E. Dummit's Math 1465 ∼ Intensive Mathematical Reasoning, Fall 2024 ∼ Homework 8, due Tue Nov 5th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each partial ordering on each set, decide whether or not the relation is a total ordering, and briefly explain your reasoning.
   (a) The alphabetical-order relation $\{(a,a),(a,b),(a,c),(b,b),(b,c),(c,c)\}$ on the set $\{a,b,c\}$.
   (b) The identity relation $\{(1,1),(2,2),(3,3),(4,4)\}$ on the set $\{1,2,3,4\}$.
   (c) The divisibility relation on the set $\{1,2,3,4,5,\dots\}$ of positive integers.
   (d) The divisibility relation on the set $\{1,10,100,1000,\dots\}$ of powers of 10.

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2. For each f, A, and B, identify whether or not f is a function from A to B.
   (a) $A=\{1,2,3\}$, $B=\{4\}$, where $f=\{(1,4),(2,4),(3,4)\}$.
   (b) $A=\{1\}$, $B=\{2,3,4\}$, where $f=\{(1,2),(1,3),(1,4)\}$.
   (c) $A=\{1,2,3\}$, $B=\{4\}$, where $f=\{(1,2),(2,3),(3,4)\}$.
   (d) $A=\{1,2,3\}$, $B=\{2,3,4\}$, where $f=\{(1,2),(2,3),(3,4)\}$.
   (e) $A=\{1,2,3\}$, $B=\{2,3,4,5,6\}$, where $f=\{(1,2),(2,3),(3,4)\}$.

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3. For each f, A, and B, identify whether or not f is a well-defined function from A to B. (Hint: Exactly three of them are well defined.)
   (a) $A=\mathbb{Q}$, $B=\mathbb{Q}$, where $f(a/b)=a/b^2$.
   (b) $A=\mathbb{Q}$, $B=\mathbb{Q}$, where $f(a/b)=a^2/b^2$.
   (c) $A=\mathbb{Z}$, $B=\mathbb{Z}/m\mathbb{Z}$, where $f(a)=\overline{a}$, with $m>1$ a fixed modulus.
   (d) $A=\mathbb{Z}/m\mathbb{Z}$, $B=\mathbb{Z}$, where $f(\overline{a})=a$, with $m>1$ a fixed modulus.
   (e) $A=\mathbb{Z}/m\mathbb{Z}$, $B=\mathbb{Z}/m\mathbb{Z}$, where $f(\overline{a})=\overline{a^2}$, with $m>1$ a fixed modulus.
   (f) $A=\mathbb{Z}/m\mathbb{Z}$, $B=\mathbb{Z}/m\mathbb{Z}$, where $f(\overline{a})=\overline{a}^{-1}$, with $m>1$ a fixed modulus.

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4. For each function $f: A \to B$, determine whether f is (i) one-to-one, (ii) onto, and (iii) a bijection.
   (a) $f_1=\{(1,4),(2,5),(3,6)\}$ from $A=\{1,2,3\}$ to $B=\{4,5,6\}$.
   (b) $f_2=\{(1,4),(2,5),(3,6)\}$ from $A=\{1,2,3\}$ to $B=\{4,5,6,7\}$.
   (c) $f_3=\{(1,5),(2,6),(3,6),(4,6)\}$ from $A=\{1,2,3,4\}$ to $B=\{5,6\}$.
   (d) $f_4(x)=2x+1$ from $A=\mathbb{R}$ to $B=\mathbb{R}$.
   (e) $f_5(n)=2n+1$ from $A=\mathbb{Z}$ to $B=\mathbb{Z}$.
   (f) $f_6(n)=\dfrac{1}{n^2+1}$ from $A=\mathbb{Z}$ to $B=\mathbb{Q}$.
   (g) $f_7(a)=\overline{a}$ from $A=\mathbb{Z}$ to $B=\mathbb{Z}/m\mathbb{Z}$, with $m>1$ a fixed modulus.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. Show the following:

   (a) Suppose R is a partial ordering on a set A. Show that R^{-1} is also a partial ordering on A.

   (b) Suppose R is a total ordering on a set A. Show that R^{-1} is also a total ordering on A.

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6. Suppose $f : A \to B$ is a function and S is an equivalence relation on B. Prove that the relation $R : A \to A$ given by $R = \{(a, b) \in A \times A \,:\, (f(a), f(b)) \in S\}$ is an equivalence relation on A.

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7. Suppose A, B, and C are sets.

   (a) If $f : B \to C$ and $g : A \to B$ are both one-to-one, prove that $f \circ g$ is also one-to-one.

   (b) If $f : B \to C$ and $g : A \to B$ are both onto, prove that $f \circ g$ is also onto.

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8. Suppose $f : A \to B$ is a function.

   - If $S \subseteq A$, we write $f(S) = \{f(s) : s \in S\}$ and call $f(S)$ the <u>image</u> of S.
   - If $T \subseteq B$, we write $f^{-1}(T) = \{a \in A : f(a) \in T\}$ and call $f^{-1}(T)$ the <u>inverse image</u> of T.
   - When $T = \{b\}$ is a single element, we write $f^{-1}(T)$ as $f^{-1}(b)$ rather than $f^{-1}(\{b\})$, with the understanding that $f^{-1}(b)$ is a set that could be empty or contain more than one element.

   Example: For the function $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = x^4$, with $A = \{1, 4\}$ we have $h(A) = \{1, 64\}$ and $h^{-1}(A) = \{-\sqrt{2}, -1, 1, \sqrt{2}\}$. We also have $h(\{-1\}) = h(\{1\}) = \{1\}$ while $h^{-1}(1) = \{1, -1\}$ and $h^{-1}(-1) = \emptyset$.

   (a) Suppose $g : \mathbb{R} \to \mathbb{R}$ is the function with $g(x) = x^2$ and recall the notation $[a, b] = \{x \in \mathbb{R} : a \le x \le b\}$ for a closed interval. Match the following ten image or inverse image sets with their values.
   <u>Sets</u>: $g(\{1, 2\})$, $g([-1, 2])$, $g([0, 1])$, $g(\emptyset)$, $g^{-1}(0)$, $g^{-1}(1)$, $g^{-1}(\{1, 4\})$, $g^{-1}(-1)$, $g^{-1}([4, 9])$, $g^{-1}([0, 1])$.
   <u>Values</u>: $\emptyset$, $\emptyset$, $\{-1, 1\}$, $[0, 4]$, $\{0\}$, $[-1, 1]$, $\{-2, -1, 1, 2\}$, $[0, 1]$, $[-3, -2] \cup [2, 3]$, $\{1, 4\}$.

   (b) Suppose $f : A \to B$ is a function. If S is any subset of A, show that $S \subseteq f^{-1}(f(S))$.

   (c) Find an example of a subset S of $\mathbb{R}$ such that $S \neq g^{-1}(g(S))$ for the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$.

   (d) Suppose $f : A \to B$ is a function. If T is any subset of B, show that $f(f^{-1}(T)) \subseteq T$.

   (e) Find an example of a subset T of $\mathbb{R}$ such that $T \neq g(g^{-1}(T))$ for the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$.

   (f) Suppose $f : A \to B$ is a function. If B_1 and B_2 are subsets of B, show $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

   (g) Find an example of subsets A_1 and A_2 of $\mathbb{R}$ such that $g(A_1 \cap A_2) \neq g(A_1) \cap g(A_2)$ for the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$.
