1. For each relation R on the given set, identify whether or not R is (i) reflexive, (ii) symmetric, (iii) transitive, and (iv) an equivalence relation.

   (a) $R_1 = \{(1,1), (2,1), (2,2)\}$ on the set $\{1,2\}$.

   - This relation is $\boxed{\text{reflexive}}$ because it contains $(1,1)$ and $(2,2)$.
   - This relation is $\boxed{\text{not symmetric}}$ because $(2,1) \in R$ but its reverse $(1,2) \notin R$.
   - This relation is $\boxed{\text{transitive}}$ by a direct calculation.
   - This relation is $\boxed{\text{not an equivalence relation}}$ because it is not symmetric.

   (b) $R_2 = \{(1,1), (2,1), (2,2)\}$ on the set $\{1,2,3\}$.

   - This relation is $\boxed{\text{not reflexive}}$ because it does not contain $(3,3)$.
   - This relation is $\boxed{\text{not symmetric}}$ because $(2,1) \in R$ but its reverse $(1,2) \notin R$.
   - This relation is $\boxed{\text{transitive}}$ by a direct calculation.
   - This relation is $\boxed{\text{not an equivalence relation}}$ because it is not reflexive and not symmetric.

   (c) $R_3 = \{(1,1), (2,2), (2,3), (3,2), (3,3)\}$ on the set $\{1,2,3\}$.

   - This relation is $\boxed{\text{reflexive}}$ because it contains $(1,1)$, $(2,2)$, and $(3,3)$.
   - This relation is $\boxed{\text{symmetric}}$ because the reverse of each pair is also in the set.
   - This relation is $\boxed{\text{transitive}}$ by a direct calculation.
   - This relation is $\boxed{\text{an equivalence relation}}$ because it is reflexive, symmetric, and transitive.

   (d) R_4, the relation on human beings where $a\ R_4\ b$ means "a has the same last name as b".

   - This relation is $\boxed{\text{reflexive}}$ because all people have the same last name as themselves.
   - This relation is $\boxed{\text{symmetric}}$ because if a and b have the same last name, so do b and a.
   - This relation is $\boxed{\text{transitive}}$ because if a, b and b, c have the same last name, all three do.
   - This relation is $\boxed{\text{an equivalence relation}}$ because it is reflexive, symmetric, and transitive.

   (e) R_5, the relation on human beings where $a\ R_5\ b$ means "a is a parent of b".

   - This relation is $\boxed{\text{not reflexive}}$ because nobody can be their own parent.
   - This relation is $\boxed{\text{not symmetric}}$ because if a is a parent of b, then b cannot be a parent of a (this would mean that a is their own grandparent, which in the absence of time travel, is not possible!).
   - This relation is $\boxed{\text{not transitive}}$ because if a is a parent of b and b is a parent of c, then a is not necessarily a parent of c (this would in fact mean a is a grandparent of c).
   - This relation is $\boxed{\text{not an equivalence relation}}$ because it satisfies none of the three requirements.

   (f) $R_6 = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 = y^2\}$ on the set of real numbers $\mathbb{R}$.

   - This relation is $\boxed{\text{reflexive}}$ because $a^2 = a^2$ for any a.
   - This relation is $\boxed{\text{symmetric}}$ because if $x^2 = y^2$ then $y^2 = x^2$.
   - This relation is $\boxed{\text{transitive}}$ because if $x^2 = y^2$ and $y^2 = z^2$ then $x^2 = z^2$.
   - This relation is $\boxed{\text{an equivalence relation}}$ because it satisfies all three requirements.

   (g) R_7, the empty relation on the empty set. (Be **very** careful with the quantifiers in the definitions!)

   - This relation is $\boxed{\text{reflexive}}$ because for all $a \in \emptyset$, it is (vacuously) true that $a\ R_7\ a$. (We can view this as a conditional statement: "if $a \in \emptyset$, then $a\ R_7\ a$": then the hypothesis is always false, so the conditional is true.)

- This relation is $\boxed{\text{symmetric}}$ because for all $a, b \in \emptyset$, it is (vacuously) true that $a\ R_7\ b$ implies $b\ R_7\ a$.
- This relation is $\boxed{\text{transitive}}$ because for all $a, b, c \in \emptyset$, it is (vacuously) true that $a\ R_7\ b$ and $b\ R_7\ c$ together imply that $a\ R_7\ c$.
- This relation is $\boxed{\text{an equivalence relation}}$ because it is reflexive, symmetric, and transitive.

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2. For each relation R on the given set, identify whether or not R is (i) reflexive, (ii) antisymmetric, (iii) transitive, (iv) a partial ordering.

(a) $R_8 = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$, on the set $\{a, b, c\}$.

- The relation is $\boxed{\text{reflexive}}$ since it contains $(a, a), (b, b), (c, c)$.
- The relation is $\boxed{\text{antisymmetric}}$ because it contains (a, b) but not (b, a), and (b, c) but not (c, b).
- The relation is $\boxed{\text{transitive}}$ as can be checked directly.
- Hence it is $\boxed{\text{a partial ordering}}$.

(b) $R_9 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 \leq y^2\}$ on the set of real numbers $\mathbb{R}$.

- The relation is $\boxed{\text{reflexive}}$ because $x^2 \leq x^2$ for every x.
- The relation is $\boxed{\text{not antisymmetric}}$, because for example $(-1)^2 \leq 1^2$ and $1^2 \leq (-1)^2$, but $-1 \neq 1$.
- The relation is $\boxed{\text{transitive}}$ since if $x^2 \leq y^2$ and $y^2 \leq z^2$ then $x^2 \leq z^2$.
- Since it is not antisymmetric, it is $\boxed{\text{not a partial ordering}}$.

(c) $R_{10} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < x\}$ on the set of real numbers $\mathbb{R}$.

- The relation is $\boxed{\text{not reflexive}}$ because $(x, x) \notin R$ for any x.
- The relation is $\boxed{\text{antisymmetric}}$ because $y < x$ and $x < y$ cannot occur at the same time, so the condition is vacuously true.
- The relation is $\boxed{\text{transitive}}$ because if $y < x$ and $z < y$ then $z < x$.
- Since it is not reflexive, it is $\boxed{\text{not a partial ordering}}$.

(d) $R_{11} = \{(4, 4), (4, 8), (4, 12), (6, 6), (6, 12), (8, 8), (10, 10), (12, 12)\}$, the divisibility relation on $\{4, 6, 8, 10, 12\}$.

- The relation is $\boxed{\text{reflexive}}$, $\boxed{\text{antisymmetric}}$, $\boxed{\text{transitive}}$, and $\boxed{\text{a partial ordering}}$ because it is a restriction of a partial ordering from a larger set. (Alternatively, one could check it directly.)

(e) $R_{12} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b = a \text{ or } b = a + 1\}$ on the set of integers $\mathbb{Z}$.

- The relation is $\boxed{\text{reflexive}}$ because all pairs (a, a) are in R.
- The relation is $\boxed{\text{antisymmetric}}$ because if (a, b) and (b, a) are in R, then $a = b$.
- The relation is $\boxed{\text{not transitive}}$ since for example $(1, 2)$ and $(2, 3)$ are in R, but $(1, 3)$ is not.
- Since it is not transitive, it is $\boxed{\text{not a partial ordering}}$.

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3. Find all partitions of the set $\{1, 2, 3\}$ and write down all ordered pairs in the corresponding equivalence relation for each.

- We simply list all of the possible partitions; there are 5 in total.
- $\mathcal{P}_1 = \{\{1\}, \{2\}, \{3\}\}$ with relation $R_1 = \{(1, 1), (2, 2), (3, 3)\}$.
- $\mathcal{P}_2 = \{\{1\}, \{2, 3\}\}$ with relation $R_2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$.
- $\mathcal{P}_3 = \{\{1, 2\}, \{3\}\}$ with relation $R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$.
- $\mathcal{P}_4 = \{\{1, 3\}, \{2\}\}$ with relation $R_4 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$.
- $\mathcal{P}_5 = \{\{1, 2, 3\}\}$ with relation $R_5 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

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4. Another property of relations that arises on occasion is as follows: we say a relation R on a set A is irreflexive when $a \not R\, a$ for all $a \in A$. This property is essentially the opposite of being reflexive.

Example: The order relation $<$ on real numbers is irreflexive, because $a < a$ is false for all real numbers a. (In fact this particular relation is one main motivation for considering irreflexive relations, since it is a property held by strict inequalities.)

(a) For each relation R_1 through R_{12} in problems 1 and 2, identify whether the relation is irreflexive.

- The only irreflexive relations are R_5 (no person can be their own parent), R_7 (vacuously since $a \in A$ it is true that $(a, a) \notin \emptyset$), and R_{10} (since $x < x$ is false for every x).

(b) Give an example of a relation that is not reflexive and also not irreflexive. (Thus, being irreflexive is *not* the same as being not reflexive.)

- As observed in (a), R_2 is not reflexive, but is also not irreflexive. In general, any relation where some but not all pairs (a, a) (for $a \in A$) are in R, will be neither reflexive nor irreflexive.

(c) Does there exist a relation on $A = \{1, 2, 3\}$ that is both reflexive and irreflexive? Does there exist any relation on any set that is both reflexive and irreflexive? Explain why or why not.

- There is no such relation on $A = \{1, 2, 3\}$. If there were, then it would have to contain $(1, 1)$ since it is reflexive, but then it would not be irreflexive.
- By this same argument, if A is nonempty, then for any $a \in A$, a reflexive irreflexive relation would have to contain (a, a), but then it would not be irreflexive. So the only possibility would be for A to be empty, in which case R would have to be the empty relation.
- But in fact, the empty relation R_7 on the empty set is reflexive and also irreflexive – this means it is in fact the only example!

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5. Suppose that R is a relation on the set A.

Proposition: If R is symmetric and transitive, then R is reflexive.
Proof: Let $a \in A$ be arbitrary. Because R is symmetric, if $a\, R\, b$ then $b\, R\, a$. Therefore, applying transitivity to $a\, R\, b$ and $b\, R\, a$ yields $a\, R\, a$. Because a was arbitrary, we conclude $a\, R\, a$ for every $a \in A$, so R is reflexive.

(a) The proof given above is erroneous. (If it were correct, we would not bother to include reflexivity in the definition of an equivalence relation!) Explain, briefly, what the error in the proof is. [Hint: See problem 8 of homework 3 for inspiration.]

- The problem is similar to the error described in problem 8 of homework 3: the proof assumes that there exists an element $b \in A$ for which $a\, R\, b$ is true, and this is not necessarily a valid assumption.
- If such an element b exists, then the argument is valid, but if no such b exists, then there is no relation statement $a\, R\, b$ to which we could apply symmetry, and thus there is no way to deduce that $a\, R\, a$.

(b) Construct a counterexample to the proposition using the set $A = \{1, 2\}$.

- Per the observation above, we want to find an example in which an element of A is not contained in any ordered pair in the relation.
- One way to do this is to take $R = \{(1, 1)\}$: then the relation is trivially symmetric and transitive, but it is not reflexive. (Similarly, $R = \{(2, 2)\}$ also works.)
- The other option is to take R to be the empty relation, which is always symmetric and transitive, but is again not reflexive here.
- Note that $R = \{(1, 2), (2, 1)\}$ is *not* transitive, since $(1, 2), (2, 1) \in R$ but $(1, 1) \notin R$.

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6. Prove that the relation $\Leftrightarrow$ on logical propositions is an equivalence relation. (This justifies our terminology of saying that $\Leftrightarrow$ indicates "logical equivalence".)

   - Each of the three properties can be shown formally with a truth table. Alternatively, we can give explanations directly as follows.
   - Reflexive: We have $P \Leftrightarrow P$ for any proposition P since the statement $P \Leftrightarrow P$ is true both when P is true and when P is false.
   - Symmetric: If $P \Leftrightarrow Q$ is true, then P, Q have the same truth value, and then $Q \Leftrightarrow P$ is also true.
   - Transitive: If $P \Leftrightarrow Q$ and $Q \Leftrightarrow R$ are both true, then P, Q and Q, R have the same truth value, and then P, R also have the same truth value, so $P \Leftrightarrow R$ is also true.

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7. Suppose $R : A \to B$ and $S : A \to B$ are relations (i.e., subsets of $A \times B$). For each statement below, identify whether it is true or false. If it is true then prove it, and if it is false then give a counterexample. [Hint: There are two true statements in total.]

   (a) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

   - This statement is $\boxed{\text{true}}$: suppose $(b, a) \in R^{-1}$, so that $(a, b) \in R$ by definition.
   - Then $(a, b) \in S$ since $R \subseteq S$, and hence $(b, a) \in S^{-1}$ by definition of S^{-1}.
   - This means $R^{-1} \subseteq S^{-1}$ as claimed.

   (b) $(R \cup S)^{-1} = R^{-1} \cap S^{-1}$.

   - This statement is $\boxed{\text{false}}$.
   - Here is a counterexample: take $R = \{(1, 1)\}$ and $S = \{(1, 2)\}$ with $A = B = \{1, 2\}$. Then $R \cup S = \{(1, 1), (1, 2)\}$ so $(R \cup S)^{-1} = \{(1, 1), (2, 1)\}$, while $R^{-1} = \{(1, 1)\}$ and $S^{-1} = \{(2, 1)\}$ so that $R^{-1} \cap S^{-1} = \emptyset$.
   - <u>Remark</u>: In fact, the correct statement is $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

   (c) $R = R^{-1}$ if and only if R is symmetric.

   - This statement is $\boxed{\text{true}}$.
   - First suppose $R = R^{-1}$. If $(a, b) \in R$, then since $(b, a) \in R^{-1}$ this means $(b, a) \in R$. In other words, $(a, b) \in R$ implies $(b, a) \in R$, meaning R is symmetric.
   - Conversely, suppose R is symmetric. If $(a, b) \in R$, then since R is symmetric, this means $(b, a) \in R$ and so $(a, b) \in R^{-1}$. Hence $R \subseteq R^{-1}$. On the other hand, if $(a, b) \in R^{-1}$ then $(b, a) \in R$, so by symmetry this means $(a, b) \in R$. Hence $R^{-1} \subseteq R$, and thus $R = R^{-1}$ as claimed.

   (d) The only relation on a set A that is both symmetric and antisymmetric is the identity relation.

   - This statement is $\boxed{\text{false}}$.
   - Here is a counterexample: take $A = \{1, 2\}$ and $R = \{(1, 1)\}$. Then R is both symmetric and antisymmetric, but is not the identity relation.
   - More generally, any proper subset of the identity relation on A is symmetric and antisymmetric.

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8. The goal of this problem is to show that taking intersections of relations preserves most of their standard properties we have defined. Suppose I is a nonempty indexing set and R_i is a relation on the set A for each $i \in I$.

   (a) If each R_i is reflexive, show that $\bigcap_{i \in I} R_i$ is also reflexive.

   - If each R_i is reflexive, then for any $a \in A$ we have $(a, a) \in R_i$. By definition of the intersection we therefore have $(a, a) \in \bigcap_{i \in I} R_i$ which means $\bigcap_{i \in I} R_i$ is reflexive.

   (b) If each R_i is symmetric, show that $\bigcap_{i \in I} R_i$ is also symmetric.

   - Suppose $(a, b) \in \bigcap_{i \in I} R_i$. By definition of the intersection we have $(a, b) \in R_i$ for each i, and so since each R_i is symmetric this means $(b, a) \in R_i$ for each i hence $(b, a) \in \bigcap_{i \in I} R_i$. Therefore $\bigcap_{i \in I} R_i$ is also symmetric.

   (c) If each R_i is antisymmetric, show that $\bigcap_{i \in I} R_i$ is also antisymmetric.

   - Suppose $(a, b), (b, a) \in \bigcap_{i \in I} R_i$. By definition of the intersection we have $(a, b), (b, a) \in R_i$ for each i, and so since each R_i is antisymmetric this means $a = b$. In fact, here we just need $(a, b), (b, a)$ to be in one of the R_i.

   (d) If each R_i is transitive, show that $\bigcap_{i \in I} R_i$ is also transitive.

   - Suppose $(a, b), (b, c) \in \bigcap_{i \in I} R_i$. By definition of the intersection we have $(a, b), (b, c) \in R_i$ for each i, and so since each R_i is transitive this means $(a, c) \in R_i$ for each i hence $(a, c) \in \bigcap_{i \in I} R_i$. Therefore $\bigcap_{i \in I} R_i$ is also transitive.

   (e) Deduce that the intersection of an arbitrary collection of equivalence relations is an equivalence relation, and that the intersection of an arbitrary collection of partial orderings is a partial ordering.

   - The result for equivalence relations follows immediately from (a) + (b) + (d), while the result for partial orderings follows immediately from (a) + (c) + (d).

   (f) If R is any relation on A, show that R has a well-defined "equivalence closure": namely, a relation $\tilde{R}$ on A such that $R \subseteq \tilde{R}$ where $\tilde{R}$ is an equivalence relation such that $\tilde{R}$ is a subset of any other equivalence relation containing R. [Hint: Take the intersection of all equivalence relations containing R. Make sure to show that this intersection is not empty.]

   - Per the hint, let $\mathcal{F} = \{R_i : R \subseteq R_i \text{ and } R_i \text{ is an equivalence relation on } A\}$ be the collection of all equivalence relations containing R. Then $\mathcal{F}$ is nonempty since $A \times A$ is an equivalence relation and contains R.
   - By (e), the intersection $\tilde{R} = \bigcap_{i \in I} R_i$ is an equivalence relation containing R. Furthermore, by the definition of the intersection we have $\tilde{R} \subseteq R_i$ for all $R_i \in \mathcal{F}$, so since $\tilde{R} \in \mathcal{F}$ as it is an equivalence relation containing R, in fact $\tilde{R}$ is the smallest element of $\mathcal{F}$ as claimed.

   (g) Illustrate (f) by finding the equivalence closures of the relations $R_1 = \{(1, 2), (1, 3), (2, 4)\}$ and $R_2 = \{(1, 2), (3, 3)\}$ on $A = \{1, 2, 3, 4\}$. [Hint: Identify which elements must go together in each equivalence class.]

   - For R_1 if $\sim$ denotes the equivalence closure, note that we must have $1 \sim 2$, $1 \sim 3$, and $2 \sim 4$, and so all of 1, 2, 3, 4 are in the same equivalence class. So the equivalence closure is just $A \times A$.
   - For R_2 in the same way we must have $1 \sim 2$ and $3 \sim 3$ but no other things are required, so the minimal choice of equivalence classes is $\{1, 2\}$, $\{3\}$, and $\{4\}$. The equivalence closure is then $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$.
