E. Dummit's Math 1465 ~ Intensive Mathematical Reasoning, Fall 2024 ~ Homework 7, due Tue Oct 29th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each relation R on the given set, identify whether or not R is (i) reflexive, (ii) symmetric, (iii) transitive, and (iv) an equivalence relation.
   (a) $R_1 = \{(1,1),\ (2,1),\ (2,2)\}$ on the set $\{1,2\}$.
   (b) $R_2 = \{(1,1),\ (2,1),\ (2,2)\}$ on the set $\{1,2,3\}$.
   (c) $R_3 = \{(1,1),\ (2,2),\ (2,3),\ (3,2),\ (3,3)\}$ on the set $\{1,2,3\}$.
   (d) R_4, the relation on human beings where $a\ R_4\ b$ means "a has the same last name as b".
   (e) R_5, the relation on human beings where $a\ R_5\ b$ means "a is a parent of b".
   (f) $R_6 = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 = y^2\}$ on the set of real numbers $\mathbb{R}$.
   (g) R_7, the empty relation on the empty set. (Be **very** careful with the quantifiers in the definitions!)

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2. For each relation R on the given set, identify whether or not R is (i) reflexive, (ii) antisymmetric, (iii) transitive, (iv) a partial ordering.
   (a) $R_8 = \{(a,a),(a,b),(a,c),(b,b),(b,c),(c,c)\}$, on the set $\{a,b,c\}$.
   (b) $R_9 = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 \le y^2\}$ on the set of real numbers $\mathbb{R}$.
   (c) $R_{10} = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y < x\}$ on the set of real numbers $\mathbb{R}$.
   (d) $R_{11} = \{(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),(10,10),(12,12)\}$, the divisibility relation on $\{4,6,8,10,12\}$.
   (e) $R_{12} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : b = a \text{ or } b = a+1\}$ on the set of integers $\mathbb{Z}$.

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3. Find all partitions of the set $\{1,2,3\}$ and write down all ordered pairs in the corresponding equivalence relation for each.

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4. Another property of relations that arises on occasion is as follows: we say a relation R on a set A is <u>irreflexive</u> when $a \not\!R\ a$ for all $a \in A$. This property is essentially the opposite of being reflexive.

   Example: The order relation $<$ on real numbers is irreflexive, because $a < a$ is false for all real numbers a. (In fact this particular relation is one main motivation for considering irreflexive relations, since it is a property held by strict inequalities.)

   (a) For each relation R_1 through R_{12} in problems 1 and 2, identify whether the relation is irreflexive.
   (b) Give an example of a relation that is not reflexive and also not irreflexive. (Thus, being irreflexive is *not* the same as being not reflexive.)
   (c) Does there exist a relation on $A = \{1,2,3\}$ that is both reflexive and irreflexive? Does there exist any relation on any set that is both reflexive and irreflexive? Explain why or why not.

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5. Suppose that R is a relation on the set A.

   Proposition: If R is symmetric and transitive, then R is reflexive.
   Proof: Let $a \in A$ be arbitrary. Because R is symmetric, if $a\ R\ b$ then $b\ R\ a$. Therefore, applying transitivity to $a\ R\ b$ and $b\ R\ a$ yields $a\ R\ a$. Because a was arbitrary, we conclude $a\ R\ a$ for every $a \in A$, so R is reflexive.

   (a) The proof given above is erroneous. (If it were correct, we would not bother to include reflexivity in the definition of an equivalence relation!) Explain, briefly, what the error in the proof is. [Hint: See problem 8 of homework 3 for inspiration.]

   (b) Construct a counterexample to the proposition using the set $A = \{1, 2\}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

6. Prove that the relation $\Leftrightarrow$ on logical propositions is an equivalence relation. (This justifies our terminology of saying that $\Leftrightarrow$ indicates "logical equivalence".)

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7. Suppose $R : A \to B$ and $S : A \to B$ are relations (i.e., subsets of $A \times B$). For each statement below, identify whether it is true or false. If it is true then prove it, and if it is false then give a counterexample. [Hint: There are two true statements in total.]

   (a) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

   (b) $(R \cup S)^{-1} = R^{-1} \cap S^{-1}$.

   (c) $R = R^{-1}$ if and only if R is symmetric.

   (d) The only relation on a set A that is both symmetric and antisymmetric is the identity relation.

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8. The goal of this problem is to show that taking intersections of relations preserves most of their standard properties. Suppose I is a nonempty indexing set and R_i is a relation on the set A for each $i \in I$.

   (a) If each R_i is reflexive, show that $\bigcap_{i \in I} R_i$ is also reflexive.

   (b) If each R_i is symmetric, show that $\bigcap_{i \in I} R_i$ is also symmetric.

   (c) If each R_i is antisymmetric, show that $\bigcap_{i \in I} R_i$ is also antisymmetric.

   (d) If each R_i is transitive, show that $\bigcap_{i \in I} R_i$ is also transitive.

   (e) Deduce that the intersection of an arbitrary collection of equivalence relations is an equivalence relation, and that the intersection of an arbitrary collection of partial orderings is a partial ordering.

   (f) If R is any relation on A, show that R has a well-defined "equivalence closure": namely, a relation $\tilde{R}$ on A such that $R \subseteq \tilde{R}$ where $\tilde{R}$ is an equivalence relation such that $\tilde{R}$ is a subset of any other equivalence relation containing R. [Hint: Take the intersection of all equivalence relations containing R. Make sure to show that this intersection is not empty.]

   (g) Illustrate (f) by finding the equivalence closures of the relations $R_1 = \{(1, 2), (1, 3), (2, 4)\}$ and $R_2 = \{(1, 2), (3, 3)\}$ on $A = \{1, 2, 3, 4\}$. [Hint: Identify which elements must go together in each equivalence class.]
