1. For each pair of integers (a, b), use the Euclidean algorithm to calculate their greatest common divisor $d = \gcd(a, b)$ AND also to find integers x and y such that $d = ax + by$. (Make sure to include the Euclidean algorithm calculations in your writeup.)

   (a) $a = 12$, $b = 44$.

   - Applying the Euclidean algorithm to $a = 12$ and $b = 44$ yields
   $$\begin{aligned} 44 &= 3 \cdot 12 + 8 \\ 12 &= 1 \cdot 8 + 4 \\ 8 &= 2 \cdot 4 \end{aligned}$$
   and thus the gcd is the last nonzero remainder of $\boxed{4}$.
   - For the linear combination, we solve for the remainders:
   $$\begin{array}{rcrcl} 8 & = & & = & 1 \cdot 44 - 3 \cdot 12 \\ 4 & = & 12 - 1 \cdot 8 & = & 4 \cdot 12 - 1 \cdot 44 \end{array}$$
   and so we see $\boxed{4 = 4 \cdot 12 - 1 \cdot 44}$ so we can take $x = 4$ and $y = -1$.

   (b) $a = 20$, $b = 107$.

   - Applying the Euclidean algorithm to $a = 12$ and $b = 44$ yields
   $$\begin{aligned} 107 &= 5 \cdot 20 + 7 \\ 20 &= 2 \cdot 7 + 6 \\ 7 &= 1 \cdot 6 + 1 \\ 6 &= 6 \cdot 1 \end{aligned}$$
   and thus the gcd is the last nonzero remainder of $\boxed{1}$.
   - For the linear combination, we solve for the remainders:
   $$\begin{array}{rcrcl} 7 & = & & = & 107 - 5 \cdot 20 \\ 6 & = & 20 - 2 \cdot 7 & = & -2 \cdot 107 + 11 \cdot 20 \\ 1 & = & 7 - 1 \cdot 6 & = & 3 \cdot 107 - 16 \cdot 20 \end{array}$$
   and so we see $\boxed{1 = 3 \cdot 107 - 16 \cdot 20}$ so we can take $x = 3$ and $y = -16$.

   (c) $a = 2023$, $b = 20234$.

   - Applying the Euclidean algorithm to $a = 2023$ and $b = 20234$ yields
   $$\begin{aligned} 20234 &= 10 \cdot 2023 + 4 \\ 2023 &= 505 \cdot 4 + 3 \\ 4 &= 1 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 \end{aligned}$$
   and so the gcd is the last nonzero remainder of $\boxed{1}$.
   - For the linear combination, we solve for the remainders:
   $$\begin{array}{rcrcl} 4 & = & & = & 1 \cdot 20234 - 10 \cdot 2023 \\ 3 & = & 2023 - 505 \cdot 4 & = & -505 \cdot 20234 + 5051 \cdot 2023 \\ 1 & = & 4 - 1 \cdot 3 & = & 506 \cdot 20234 - 5061 \cdot 2023 \end{array}$$
   and so we see $\boxed{1 = -505 \cdot 20234 + 5051 \cdot 2023}$ so we can take $x = -505$ and $y = 5051$.

   (d) $a = 5567$, $b = 12445$.

- Applying the Euclidean algorithm to $a = 5567$ and $b = 12445$ yields

$$\begin{aligned} 12445 &= 2 \cdot 5567 + 1311 \\ 5567 &= 4 \cdot 1311 + 323 \\ 1311 &= 4 \cdot 323 + 19 \\ 323 &= 17 \cdot 19 \end{aligned}$$

  and so the gcd is the last nonzero remainder of $\boxed{19}$.
- For the linear combination, we solve for the remainders:

$$\begin{array}{rcccl} 1311 & = & & = & 1 \cdot 12445 - 2 \cdot 5567 \\ 323 & = & 5567 - 4 \cdot 1311 & = & -4 \cdot 12445 + 9 \cdot 5567 \\ 19 & = & 1311 - 4 \cdot 323 & = & 17 \cdot 12445 - 38 \cdot 5567 \end{array}$$

  and so we see $\boxed{19 = 17 \cdot 12445 - 38 \cdot 5567}$ so we can take $x = 17$ and $y = -38$.

(e) $a = 233$, $b = 144$.

- Applying the Euclidean algorithm to $a = 233$ and $b = 144$ yields

$$\begin{aligned} 233 &= 1 \cdot 144 + 89 \\ 144 &= 1 \cdot 89 + 55 \\ 89 &= 1 \cdot 55 + 34 \\ 55 &= 1 \cdot 34 + 21 \\ 34 &= 1 \cdot 21 + 13 \\ 21 &= 1 \cdot 13 + 8 \\ 13 &= 1 \cdot 8 + 5 \\ 8 &= 1 \cdot 5 + 3 \\ 5 &= 1 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 \end{aligned}$$

  and so the gcd is the last nonzero remainder of $\boxed{1}$.
- For the linear combination, we solve for the remainders:

$$\begin{array}{rcccl} 89 & = & & = & 233 - 1 \cdot 144 \\ 55 & = & 144 - 1 \cdot 89 & = & -1 \cdot 233 + 2 \cdot 144 \\ 34 & = & 89 - 1 \cdot 55 & = & 2 \cdot 233 - 3 \cdot 144 \\ 21 & = & 55 - 1 \cdot 34 & = & -3 \cdot 233 + 5 \cdot 144 \\ 13 & = & 34 - 1 \cdot 21 & = & 5 \cdot 233 - 8 \cdot 144 \\ 8 & = & 21 - 1 \cdot 13 & = & -8 \cdot 233 + 13 \cdot 144 \\ 5 & = & 13 - 1 \cdot 8 & = & 13 \cdot 233 - 21 \cdot 144 \\ 3 & = & 8 - 1 \cdot 5 & = & -21 \cdot 233 + 34 \cdot 144 \\ 2 & = & 5 - 1 \cdot 3 & = & 34 \cdot 233 - 55 \cdot 144 \\ 1 & = & 3 - 1 \cdot 2 & = & -55 \cdot 233 + 89 \cdot 144 \end{array}$$

  and so we see $\boxed{1 = -55 \cdot 233 + 89 \cdot 144}$ so we can take $x = 89$ and $y = 144$.

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2. Find the following:

(a) Find the gcd and lcm of 144 and 300.

- By the Euclidean algorithm, $\gcd(144, 300) = \boxed{12}$. Then $\operatorname{lcm}(144, 300) = 144 \cdot 300/12 = \boxed{3600}$.

(b) Find the gcd and lcm of $2^8 3^{11} 5^7 7^8 11^2$ and $2^4 3^8 5^7 7^7 11^{11}$.

- From the prime factorizations, the gcd is $\boxed{2^4 3^8 5^7 7^7 11^2}$ and the lcm is $\boxed{2^8 3^{11} 5^7 7^8 11^{11}}$.

(c) Find the prime factorizations of 1600, 2024, 2025, and 2024^{2025}.

- We have $1600 = \boxed{2^6 5^2}$, $2024 = \boxed{2^3 \cdot 11 \cdot 23}$, $2025 = \boxed{3^4 \cdot 5^2}$ and so $2024^{2025} = \boxed{2^{6075} 11^{2025} 23^{2025}}$.

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3. Suppose that a, b, c are integers such that $\gcd(a, b) = 1$ and that $a|c$ and $b|c$. Prove that $(ab)|c$.

- Since $a|c$ there exists an integer k such that $c = ka$. Then $b|c$ says that $b|(ka)$.
- But a and b are relatively prime, so since $b|(ka)$, the relatively prime divisibility property implies $b|k$.
- Thus, there exists an integer l such that $k = lb$. Then $c = ka = lba$, and so $ab|c$.

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4. The goal of this problem is to demonstrate that the uniqueness of prime factorizations is not as obvious as it may seem. Let S be a nonempty set of positive integers, and define an $\underline{S\text{-prime}}$ to be an element $p \in S$ such that $p > 1$ and there do not exist $a, b \in S$ such that $ab = p$ and $1 < a, b < p$. (If S is the set of all positive integers, then this definition reduces to the usual one for prime numbers.) Let $E = \{2, 4, 6, 8, 10, \dots\}$ be the set of even positive integers and $O = \{1, 3, 5, 7, 9, 11, \dots\}$ be the set of odd positive integers.

(a) Which of 2, 4, 6, 8, 10, 12, 14, and 16 are E-primes?

- We have $4 = 2 \cdot 2$, $8 = 2 \cdot 4$, $12 = 2 \cdot 6$, and $16 = 2 \cdot 8$ so these elements are not E-primes.
- On the other hand, we cannot factor 2, 6, 10, or 14 as the product of two elements of E, since the product of two elements of E is always divisible by 4. So these elements are E-primes.

(b) Show that $2n \in E$ is an E-prime if and only if n is odd. [Hint: Show the contrapositive.]

- Suppose n is even. Then $4n = 2 \cdot 2n$ is a factorization of $4n$ as the product of two elements in E, so $2n$ is not an E-prime.
- On the other hand, suppose $2n$ is not an E-prime. Then $2n = (2a)(2b) = 4ab$ for some integers a, b, so $2n$ is a multiple of 4 hence n is even.

(c) Show that 60 has two different factorizations as a product of E-primes. Deduce that E does not have unique E-prime factorization.

- We have $60 = 6 \cdot 10 = 2 \cdot 30$, and by (b) each of 2, 6, 10, and 30 is an E-prime. Since the terms are actually different, and not just rearranged, we see that the factorizations are different, and so E does not have unique E-prime factorization.

(d) Which of 1, 3, 5, 7, 9, 11, 13, and 15 are O-primes?

- We have $9 = 3 \cdot 3$ and $15 = 3 \cdot 5$ so these elements are not O-primes, and by definition 1 is also not an O-prime.
- On the other hand, as the product of two elements of E, since the product of two elements of E is always divisible by 4. So these elements are E-primes.

(e) Show that $p \in O$ is an O-prime if and only if p is an odd prime integer.

- Suppose p is an O-prime: then $p > 1$ is odd. If $p = ab$ for some positive integers a and b with $1 < a, b < 1$, then a and b must both be odd. But this would mean $a, b \in O$, which contradicts the assumption that p is an O-prime. So p must be an odd prime integer.
- Conversely, suppose p is an odd prime. Then $p \in O$ and p cannot be factored as the product of two smaller integers, so in particular p cannot be factored as the product of two smaller integers in O. So p is an O-prime.

(f) Explain why O has unique O-prime factorization.

- By (e) the O-primes are the same as the odd primes, and since we have unique prime factorization in $\mathbb{Z}$, the same proof shows that we have unique prime factorization in O.

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5. The Fibonacci numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. The first few terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

   (a) Prove that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ for every positive integer n. [Hint: Use induction.]

   - We prove the result by induction on n.
   - For the base case $n = 1$, we must verify $F_1 = F_3 - 2$, which is true because $F_3 = 3$ and $F_1 = 1$.
   - For the inductive step, we assume that $F_1 + F_2 + F_3 + \cdots + F_k = F_{k+2} - 1$ and must show that $F_1 + F_2 + F_3 + \cdots + F_k + F_{k+1} = F_{k+3} - 1$.
   - Then $F_1 + F_2 + F_3 + \cdots + F_k + F_{k+1} = [F_1 + F_2 + F_3 + \cdots + F_k] + F_{k+1} = F_{k+2} - 1 + F_{k+1} = F_{k+3} - 1$ as required.
   - Hence by induction, $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ for every positive integer n.

   (b) Prove that $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$ for every positive integer n.

   - We prove the result by induction on n.
   - For the base case $n = 1$, we must verify $F_1^2 = F_1 F_2$, which is true because $F_1 = F_2 = 2$.
   - For the inductive step, we assume that $F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2 = F_k F_{k+1}$ and must show that $F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2 + F_{k+1}^2 = F_{k+1} F_{k+2}$.
   - We have $F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2 + F_{k+1}^2 = [F_1^2 + F_2^2 + F_3^2 + \cdots + F_k^2] + F_{k+1}^2 = F_k F_{k+1} + F_{k+1}^2 = [F_k + F_{k+1}] F_{k+1} = F_{k+1} F_{k+2}$ as required.
   - Hence by induction, $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$ for every positive integer n.

   (c) Prove that $F_{n+3} - F_n$ is even for every positive integer n.

   - We prove the result by strong induction on n.
   - For the base cases we take $n = 1$ and $n = 2$. For $n = 1$ we have $F_4 - F_1 = 3 - 1 = 2$ is even, and for $n = 2$ we have $F_5 - F_2 = 5 - 1 = 4$ is also even.
   - For the inductive step, now suppose $n \geq 3$ and both $F_{n+3} - F_n$ and $F_{n+2} - F_{n-1}$ are even.
   - Then $F_{n+4} - F_{n+1} = (F_{n+3} + F_{n+2}) - (F_n + F_{n-1}) = (F_{n+3} - F_n) + (F_{n+2} - F_{n-1})$ is the sum of two even numbers hence is also even, as desired.
   - Hence by induction, $F_{n+3} - F_n$ is even for every positive integer n.

   (d) A "lyrical pattern" consists of a sequence of long and short beats, where a long beat is twice as long as a short beat. Some examples are long-long-short-long (length 7) and short-short-short-short-long (length 6). Prove that for all $n \geq 1$, the number of lyrical patterns whose length equals n short beats is the Fibonacci number F_{n+1}. [Hint: What happens if you delete the last beat in a sequence of length n?]

   - We prove the result by strong induction on n.
   - For the base cases we take $n = 1$ and $n = 2$. There is one pattern for $n = 1$ (short) and two for $n = 2$ (short-short, long), and indeed $F_2 = 1$ and $F_3 = 2$.
   - For the inductive step, now suppose $n \geq 3$ and that the result holds for patterns of total length $n-2$ and $n - 1$.
   - Suppose we have a lyrical pattern of length n. Then it either ends in a short beat or a long beat. If it ends in a short beat, deleting the last beat yields a sequence of length $n - 1$, and by hypothesis there are F_n such sequences. If it ends in a long beat, deleting the last beat yields a sequence of length $n - 2$, and by hypothesis there are F_{n-1} such sequences.
   - Since these cases don't overlap, in total there are $F_{n-1} + F_n = F_{n+1}$ lyrical patterns of length n. This establishes the inductive hypothesis so we are done.

Remark: The study of lyrical patterns by Indian poets writing in Sanskrit (e.g., Pingala in approximately 200 BCE) is the first known analysis of the Fibonacci numbers (historically so called following Fibonacci's description of them in 1202 CE, but Virahanka was the first to give a clear description of them in approximately the year 700 CE). There are very many identities involving the Fibonacci numbers, and they show up in many applications.

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6. Prove that $\log_3 5$ is irrational. [Hint: Suppose otherwise, so that $\log_3 5 = a/b$. Convert this to statement about positive integers and find a contradiction.]

- Following the hint, suppose $\log_3 5 = a/b$ for positive integers a and b.
- Exponentiating with the base 3, this means that $5 = 3^{a/b}$.
- If we then take the bth power of both sides, this yields $5^b = 3^a$.
- However, this is impossible, because by the uniqueness of prime factorizations, we cannot have $5^b = 3^a$ for any pair of positive integers (a, b): otherwise, the positive integer $n = 5^b = 3^a$ would have two different prime factorizations.
- This is a contradiction, so there cannot exist any positive integers a and b with $\log_3 5$. Thus $\log_3 5$ is irrational as claimed.

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7. The goal of this problem is to prove the rational root test from algebra, and derive some of its consequences.

(a) Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial with integer coefficients, meaning that $a_n, a_{n-1}, \ldots, a_0$ are integers. Prove the rational root test: if r/s is a rational root in lowest terms, meaning that r, s are relatively prime and $p(r/s) = 0$, then $r|a_0$ and $s|a_n$. [Hint: Clear denominators and rearrange to show that $s|a_n r^n$ and $r|a_0 s^n$.]

- If r/s is a root of $p(x)$, then $a_n(r/s)^n + a_{n-1}(r/s)^{n-1} + \cdots + a_0 = 0$. Clearing denominators yields $a_n r^n + a_{n-1}r^{n-1}s + \cdots + a_1 r s^{n-1} + a_0 s^n = 0$.
- By rearranging we see that $a_n r^n = s(-a_{n-1}r^{n-1} - \cdots - a_0 s^{n-1})$, so s divides $a_n r^n$. But since s and r are relatively prime, this means s divides a_n.
- In a similar way, since $a_0 s^n = r(-a_n r^{n-1} - \cdots - a_1 s^{n-1})$, we see that r divides $a_0 s^n$ hence a_0.

(b) Suppose x is such that $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ for some integers $a_{n-1}, \ldots, a_0$. Show that if x is not an integer, then x is irrational.

- We show the contrapositive: if x is rational, then x is an integer.
- So suppose $x = r/s$ is rational and $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ for some integers $a_{n-1}, \ldots, a_0$.
- By part (a), $r|a_0$ and $s|1$. But since $s|1$, we have $s = \pm 1$ and so $x = \pm r$ is actually an integer, as required.

(c) If n is not a perfect square, prove that $\sqrt{n}$ is irrational.

- Let $x = \sqrt{n}$: then $x^2 - n = 0$. If x were an integer k, then we would have $n = k^2$, but n is not a perfect square.
- Therefore, x is not an integer. Therefore by part (b), since x is not an integer, it is irrational.

(d) Prove that $\sqrt{2}+\sqrt{3}$ is irrational. [Hint: Show $\sqrt{2}+\sqrt{3}$ is not an integer, then consider $[(\sqrt{2}+\sqrt{3})^2 - 5]^2$.]

- We first note that $\sqrt{2} + \sqrt{3}$ is not an integer, since $1.4 < \sqrt{2} < 1.5$ and $1.7 < \sqrt{3} < 1.8$, so $3.1 < \sqrt{2} + \sqrt{3} < 3.3$.
- Now we search for a polynomial with integer coefficients of which $x = \sqrt{2} + \sqrt{3}$ is a root. Since $x^2 - 5 = 2\sqrt{6}$ this means $(x^2 - 5)^2 = 24$, which when expanded yields $x^4 - 10x^2 + 25 = 24$ so that $x^4 - 10x^2 + 1 = 0$. This means x is a root of a polynomial with integer coefficients.
- But then by part (b), since $x^4 - 10x^2 + 1 = 0$ but x is not an integer, we conclude x is irrational.
