1. Let $I = \{1, 2, 3, \dots\}$ be the set of positive integers.
   (a) For each $i \geq 1$ let F_i be the set of positive integer divisors of i (so for example, $F_6 = \{1, 2, 3, 6\}$). Find $\bigcup_{i \in I} F_i$ and $\bigcap_{i \in I} F_i$.
   - Since each F_i consists of positive integers, the union and intersection also contain only positive integers. But since $i \in F_i$ for each i, we see that every positive integer is in $\bigcup_{i \in I} F_i$, meaning that $\bigcup_{i \in I} F_i = \boxed{\mathbb{Z}_+ = \{1, 2, 3, 4, 5, \dots\}}$.
   - On the other hand, since $F_1 = \{1\}$ and $1 \in F_i$ for all $i \geq 1$, the only positive integer in all of the F_i is 1, meaning that $\bigcap_{i \in I} F_i = \boxed{\{1\}}$.

   (b) For each $i \geq 1$ let G_i be the set of positive integer multiples of i (so for example, $G_2 = \{2, 4, 6, 8, \dots\}$). Find $\bigcup_{i \in I} G_i$ and $\bigcap_{i \in I} G_i$.
   - Since each G_i consists of positive integers, the union and intersection also contain only positive integers. But since $i \in G_i$ for each i, we see that every positive integer is in $\bigcup_{i \in I} G_i$, meaning that $\bigcup_{i \in I} G_i = \boxed{\mathbb{Z}_+ = \{1, 2, 3, 4, 5, \dots\}}$.
   - On the other hand, for any fixed positive integer n, $n \notin G_{n+1}$. So no positive integer is in all of the G_i, meaning that $\bigcap_{i \in I} G_i = \boxed{\emptyset}$.

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2. Each item below contains a proposition (which may be true or may be false) and an *incorrect* proof of the proposition. Identify at least one mistake in each claimed proof:
   (a) Proposition: 1 is the largest positive integer.
   Proof: Let n be the largest positive integer. Since $n \geq 1$ that means $n^2 \geq n$. But n is the largest positive integer, so $n^2 \leq n$. We conclude that $n^2 = n$, so since n is not zero, we have $n = 1$.
   - The mistake here is that the proof assumes without justification that a largest positive integer exists. Of course, there is no largest positive integer, so that assumption is erroneous.
   - This error may seem trivial, but it's actually very important to be aware of it, because the same kind of mistake occurs all the time in optimization problems: namely, starting with an extremal element and working out a bunch of its properties, but without actually showing that such an element must exist.

   (b) Proposition: For any integer $a > 0$, there exists a unique integer $b > 0$ such that $a = b^2$.
   Proof: Suppose that there exist two values b_1 and b_2 such that $b_1^2 = a$ and $b_2^2 = a$ with $b_1 > 0$ and $b_2 > 0$. Subtracting yields $b_1^2 - b_2^2 = 0$ so that $(b_1 - b_2)(b_1 + b_2) = 0$ so that $b_1 = b_2$ or $b_1 = -b_2$. But since $b_1 > 0$ and $b_2 > 0$ we cannot have $b_1 = -b_2$. Therefore $b_1 = b_2$ which means there exists a unique integer b such that $b > 0$ and $a = b^2$, as claimed.
   - The issue is that, although the proof correctly shows that such a value b would necessarily be unique, it has not shown that b actually exists (which is part of the "there exists a unique" requirement).
   - In fact, for many values of a, there exists no such b: for example, $a = 2$.

   (c) Proposition: If $a_1 = 1$ and $a_{n+1} = 2a_n - 1$ for all $n \geq 1$, then $a_n = 2^n + 1$ for all n.
   Proof: We use induction on n. The base case $n = 1$ is trivial. For the inductive step, suppose $a_n = 2^n + 1$. Then $a_{n+1} = 2a_n - 1 = 2(2^n + 1) - 1 = 2^{n+1} + 1$ as required.
   - The argument for the inductive step is completely correct. The issue is that the base case is wrong: although $a_1 = 1$, the formula gives instead $a_1 = 2^1 + 1 = 3$.
   - The issue is that the base case is simply asserted rather than actually proven. (Of course, this would have been very obvious if the calculations for the base case were actually given in the proof!)

   (d) Proposition: If $a_1 = 2$, and $a_{n+1} = 4a_n - 4a_{n-1}$ for all $n \geq 1$, then $a_n = 2^n$ for all n.
   Proof: We use strong induction on n. The base case $n = 1$ follows since $a_1 = 2 = 2^1$. For the inductive step, suppose $a_k = 2^k$ for all $k \leq n$. Then $a_{k+1} = 4a_n - 4a_{n-1} = 4 \cdot 2^n - 4 \cdot 2^{n-1} = 4 \cdot 2^n - 2 \cdot 2^n = 2 \cdot 2^n = 2^{n+1}$ as required.

- The issue here is that the inductive step uses the two previous cases $k = n$ and $k = n - 1$, but only one base case is actually established.
- One way to see that this is a problem is to use the recurrence to find a_2 (i.e., by setting $n = 1$), it yields $a_2 = 4a_1 - 4a_0$, but a_0 has not been defined!

(e) Proposition: We have $2^n = 1$ for every nonnegative integer n.
Proof: We use strong induction on n. The base case $n = 0$ holds since $2^0 = 1$. For the inductive step suppose that $2^k = 1$ for all $k \leq n$. Then applying the inductive hypothesis yields $2^{n+1} = 2^n 2^n / 2^{n-1} = 1 \cdot 1/1 = 1$, as desired.

- The error, like in (d), is that only one base case is established, but the argument in the inductive step uses the two previous cases n and $n - 1$. That is not valid since not enough base cases have been established. (Explicitly, it fails when $n = 1$, since the case $n = -1$ was not established.)

(f) Proposition: All horses are the same color.
Proof: We use induction on n, the number of horses. The base case $n = 1$ is trivial because any 1 horse is the same color as itself. For the inductive step, suppose that any $n + 1$ horses are the same color. Ignoring the last horse yields means that we need to show that n horses are the same color, which is true by the induction hypothesis. Therefore the result holds by induction.

- The error is that the proof of the inductive step assumes $P(n + 1)$ and uses it to establish $P(n)$. This is backwards from the correct logic, which is to show that $P(n)$ implies $P(n + 1)$.

(g) Proposition: For every positive integer n, $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$.
Proof: We use induction on n. The base case $n = 1$ follows because $1 = \frac{1}{2}(1)(2)$. To show the inductive step, we want $1 + 2 + 3 + \cdots + n + (n + 1) = \frac{1}{2}(n + 1)(n + 2)$. Subtracting $n + 1$ from both sides yields $1 + 2 + 3 + \cdots + n = \frac{1}{2}(n + 1)(n + 2) - (n + 1) = \frac{1}{2}n(n + 1)$ which is true by the induction hypothesis. Therefore the result holds by induction.

- The error is the same mistake as in part (e): the inductive step starts out by assuming $P(n+1)$ and then reduces it to $P(n)$ which is true. This is backwards from the correct logic, which is to show that $P(n)$ implies $P(n + 1)$.
- In this case the mistake can be fixed by writing the steps in the correct order.

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3. Prove the following properties of divisibility:

(a) If a, b are integers then $a|b$ if and only if $(-a)|b$.

- First suppose $a|b$ so that $b = pa$ for some integer p. Then $b = (-p)(-a)$ so $(-a)|b$.
- Conversely, suppose $(-a)|b$ so that $b = q(-a)$ for some integer q. Then $b = (-q)a$ so $a|b$.

(b) If a, b, c, x, y are integers with $a|b$ and $a|c$, show that $a|(xb + yc)$.

- By definition, if $a|b$ and $a|c$, then there exist integers p and q with $b = pa$ and $c = qa$.
- Then $xb + yc = x(pa) + y(qa) = (xp)a + (yq)a = (xp + yq)a$, and so for $k = xy + yq$ we see that $xb + yc = ka$, meaning that $a|(xb + yc)$.

(c) If a, b, m are integers with $m \neq 0$, show that $a|b$ if and only if $(ma)|(mb)$.

- First suppose $a|b$, so that $b = pa$ for some integer p. Then $mb = mpa = p(ma)$ so $(ma)|(mb)$.
- Conversely suppose $(ma)|(mb)$, so that $mb = p(ma)$ for some integer p. Since $m \neq 0$ we can cancel m to conclude that $b = pa$, meaning $a|b$ as required.

(d) If a, b are integers with $a|b$ and $b|a$ then $a = b$ or $a = -b$. [Hint: Use the fact that the only divisors of 1 are 1 and -1. Be careful when $a = 0$ or $b = 0$.]

- Suppose $a|b$ and $b|a$ so that $b = pa$ and $a = qb$ for some integers p and q.
- Then $ab = pqab$ yielding $ab(pq - 1) = 0$, so either $pq = 1$ or $a = 0$ or $b = 0$. If $a = 0$ then since $a|b$ we must have $b = 0$ so since $a = b$ the result holds.
- Otherwise we have $pq = 1$ so (per the hint) we must have $q = 1$ or $q = -1$, meaning that $a = b$ or $a = -b$.
- So in all cases we have $a = b$ or $a = -b$ as required.

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4. Show the following:

(a) For all positive integers n, show that the sum $1^2 + 2^2 + 3^2 + \cdots + n^2$ equals $\dfrac{n(n+1)(2n+1)}{6}$.

- We prove this by induction on n.
- For the base case $n = 1$, we must show that $1^2 = \dfrac{1 \cdot 2 \cdot 3}{6}$ which is clearly true.
- For the inductive step, we are given that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ and must show that $1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \dfrac{(n+1)(n+2)(2n+3)}{6}$.
- By the inductive hypothesis, we can write

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 &= [1^2 + 2^2 + 3^2 + \cdots + n^2] + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{2n^3 + 3n^2 + n}{6} + (n^2 + 2n + 1) \\ &= \frac{2n^3 + 9n^2 + 13n + 6}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

  and therefore we see $1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \dfrac{(n+1)(n+2)(2n+3)}{6}$, as required.
- By induction, $1^2 + 2^2 + 3^2 + \cdots + n^2$ equals $\dfrac{n(n+1)(2n+1)}{6}$ for every positive integer n.

(b) For all positive integers n, show that the sum $3^0 + 3^1 + 3^2 + \cdots + 3^n$ equals $\dfrac{3^{n+1} - 1}{2}$.

- We prove this by induction on n.
- For the base case $n = 1$, we must show that $2^0 = \dfrac{3^1 - 1}{2}$ which is clearly true.
- For the inductive step, we suppose that $3^0 + 3^1 + 3^2 + \cdots + 3^n = \dfrac{3^{n+1} - 1}{2}$ and must show that $3^0 + 3^1 + 3^2 + \cdots + 3^n + 3^{n+1} = \dfrac{3^{n+2} - 1}{2}$.
- By the inductive hypothesis, we can write

$$\begin{aligned} 3^0 + 3^1 + 3^2 + \cdots + 3^n + 3^{n+1} &= [3^0 + 3^1 + 3^2 + \cdots + 3^n] + 3^{n+1} \\ &= \frac{3^{n+1} - 1}{2} + 3^{n+1} \\ &= \frac{3 \cdot 3^{n+1} - 1}{2} = \frac{3^{n+1} - 1}{2} \end{aligned}$$

  and therefore we see $3^0 + 3^1 + 3^2 + \cdots + 3^n + 3^{n+1} = \dfrac{3^{n+2} - 1}{2}$, as required.
- By induction, $3^0 + 3^1 + 3^2 + \cdots + 3^n$ equals $\dfrac{3^{n+1} - 1}{2}$ for every positive integer n.

(c) The sequence $\{d_n\}_{n \geq 1}$ is defined recursively by $d_1 = 3$, $d_2 = 9$, and for all $n \geq 3$, $d_n = 2d_{n-1} + 3d_{n-2}$. Prove that $d_n = 3^n$ for all positive integers n.

- We prove this by strong induction on n.
- For the base cases $n = 1$ and $n = 2$, we see $d_1 = 3 = 3^1$ and $d_2 = 9 = 3^2$ as required.
- For the inductive step, we suppose that $d_n = 3^n$ and $d_{n-1} = 3^{n-1}$ for some $n \geq 3$, and must show that $d_{n+1} = 3^{n+1}$.

- By the inductive hypotheses and the definition of d_{n+1}, we can write

$$\begin{aligned} d_{n+1} &= 2d_n + 3d_{n-1} \\ &= 2 \cdot 3^n + 3 \cdot 3^{n-1} \\ &= 2 \cdot 3^n + 3^n = 3 \cdot 3^n = 3^{n+1} \end{aligned}$$

  as required.
- By induction, $d_n = 3^n$ for all positive integers n.

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5. Recall that an integer n is even if $n = 2a$ for some integer a, while an integer n is odd if $n = 2b + 1$ for some integer b. As noted in class, as a consequence of the division algorithm, every integer is either even or odd, and no integer is both.

   (a) Show that the sum of two even or two odd integers is even, and that the sum of an even integer and an odd integer is odd.

   - If m and n are both even then $m = 2a$ and $n = 2b$ for some integers a and b, in which case $m + n = 2a + 2b = 2(a + b)$ is also even.
   - If m and n are both odd then $m = 2a + 1$ and $n = 2b + 1$ for some integers a and b, in which case $m + n = 2a + 1 + 2b + 1 = 2(a + b + 1)$ is even.
   - If m is even and n is odd then $m = 2a$ and $n = 2b + 1$ for some integers a and b, in which case $m + n = 2a + 2b + 1 = 2(a + b) + 1$ is odd. The same argument applies if m is odd and n is even.

   (b) Show that the product of an even integer with any integer is even, and the product of two odd integers is odd.

   - If m is even then $m = 2a$ for some integer a. Then for any even integer n, we have $mn = 2an = 2(an)$, so mn is even. By the same argument, if n is even then mn is also even.
   - If m and n are both odd then $m = 2a + 1$ and $n = 2b + 1$ for some integers a and b, in which case $mn = (2a + 1)(2b + 1) = 2(2ab + a + b) + 1$ is odd.

   (c) Show that if n is even then n^2 is even, and if n is odd then n^2 is odd.

   - If n is even, then $n^2 = n \cdot n$ is even by part (b). This is the forward direction of the biconditional.
   - If n is odd, then $n^2 = n \cdot n$ is odd, also by part (b).

   (d) Deduce that n^2 is even if and only if n is even. [Hint: What is the contrapositive of "if n is odd then n^2 is odd"?]

   - This is an if-and-only-if statement so we must prove both directions: if n is even then n^2 is even, and if n^2 is even then n is even.
   - The first statement was done in (c). In (c) we also showed that if n is odd then n^2 is odd.
   - But the contrapositive of this last statement is "if n^2 is even, then n is even" (since an integer that is not odd is even), which is the other statement. So we are done.

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6. The goal of this problem is to prove the Binomial Theorem. First, we define the factorial function as $n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$, so that for example $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and so forth: in general, $n! = n \cdot (n-1)!$ for $n \geq 1$. Now define the binomial coefficient $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ for integers $0 \leq k \leq n$, and note that $\binom{n}{0} = \binom{n}{n} = 1$ for every n.

   (a) Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for every $0 \leq k \leq n$. Conclude in particular that $\binom{n}{k}$ is always an integer.

   - We have $\binom{n}{k} = n \cdot \dfrac{(n-1)!}{k!(n-k)!} = (n-k) \cdot \dfrac{(n-1)!}{k!(n-k)!} + k \cdot \dfrac{(n-1)!}{k!(n-k)!} = \dfrac{(n-1)!}{k!(n-k-1)!} + \dfrac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

- Then we can induct on n to see $\binom{n}{k}$ is always an integer: the base cases $n = 0$ and $n = 1$ are obvious. For the inductive step, observe that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ is the sum of two integers for any value of k with $1 \le k \le n-1$, and $\binom{n}{0}$ and $\binom{n}{n}$ are also integers.

(b) Suppose that x and y are arbitrary real numbers. Prove the Binomial Theorem: that $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k$ for any positive integer n. [Hint: Induct on n.]

- We use induction on n. The base case $n = 1$ is obvious, since $x + y = x + y$.
- For the inductive step, observe that

$$\begin{aligned}
(x+y)^n &= (x+y)\cdot(x+y)^{n-1} \\
&= (x+y)\sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-1-k}y^k \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{j=0}^{n-1}\binom{n-1}{j}x^{n-1-j}y^{j+1} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{n-k}y^k + \sum_{k=0}^{n-1}\binom{n-1}{k-1}x^{n-k}y^k \\
&= \sum_{k=0}^{n-1}\left[\binom{n-1}{k} + \binom{n-1}{k-1}\right]x^{n-k}y^k = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k
\end{aligned}$$

where we made the substitution $j = k - 1$ in the third equation, and used the result of part (a) in the final step.

Remark: The binomial coefficient $\binom{n}{k}$ counts the number of ways of selecting a subset of k elements from the set $\{1, 2, 3, \ldots, n\}$. Intuitively, for (b), in distributing out the product $(x + y)^n = (x + y)(x + y)\cdots(x + y)$, a term $x^{n-k}y^k$ is formed when we select a y from exactly k of the terms. So the total number of ways to obtain a term $x^{n-k}y^k$ is the same as the number of ways of selecting a subset of k elements from $\{1, 2, 3, \ldots, n\}$, and there are $\binom{n}{k}$ such subsets.

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7. Suppose that $\mathcal{A} = \{A_i : i \in I\}$ and $\mathcal{B} = \{B_j : j \in J\}$ are two families of sets indexed by I and J respectively.

(a) Prove that for each $s \in I$ it is true that $\bigcap_{i\in I} A_i \subseteq A_s$.

- This is essentially by definition. Explicitly, for any $x \in \bigcap_{i\in I} A_i$ we have $x \in A_i$ for all $i \in I$. In particular, we have $x \in A_s$. Thus $x \in \bigcap_{i\in I} A_i$ implies $x \in A_s$, which means $\bigcap_{i\in I} A_i \subseteq A_s$ as claimed.

(b) Prove that for each $t \in J$ it is true that $B_t \subseteq \bigcup_{j\in J} B_j$.

- This is very similar to (a). Explicitly, $\bigcup_{j\in J} B_j$ consists of all elements x such that $x \in B_j$ for some $j \in J$. So in particular, if $x \in B_t$ then $x \in \bigcup_{j\in J} B_j$. Thus $x \in B_t$ implies $x \in \bigcup_{j\in J} B_j$, which means $B_t \subseteq \bigcup_{j\in J} B_j$ as claimed.

(c) Prove that if $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ then $\bigcap_{i\in I} A_i \subseteq \bigcup_{j\in J} B_j$. [Hint: Suppose $S \in \mathcal{A} \cap \mathcal{B}$ and apply (a) and (b) to it.]

- Following the hint suppose that $S \in \mathcal{A} \cap \mathcal{B}$. Then $S = A_s$ for some $s \in I$ and also $S = B_t$ for some $t \in J$.
- By (a) and (b) we then have $\bigcap_{i\in I} A_i \subseteq A_s = S = B_t \subseteq \bigcup_{j\in J} B_j$, and so $\bigcap_{i\in I} A_i \subseteq \bigcup_{j\in J} B_j$ as claimed.
