E. Dummit's Math 1465 ∼ Intensive Mathematical Reasoning, Fall 2024 ∼ Homework 4, due Tue Oct 1st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $I = \{1, 2, 3, \dots\}$ be the set of positive integers.

   (a) For each $i \geq 1$ let F_i be the set of positive integer divisors of i (so for example, $F_6 = \{1, 2, 3, 6\}$). Find $\bigcup_{i \in I} F_i$ and $\bigcap_{i \in I} F_i$.

   (b) For each $i \geq 1$ let G_i be the set of positive integer multiples of i (so for example, $G_2 = \{2, 4, 6, 8, \dots\}$). Find $\bigcup_{i \in I} G_i$ and $\bigcap_{i \in I} G_i$.

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2. Each item below contains a proposition (which may be true or may be false) and an *incorrect* proof of the proposition. Identify at least one mistake in each claimed proof:

   (a) Proposition: 1 is the largest positive integer.
   Proof: Let n be the largest positive integer. Since $n \geq 1$ that means $n^2 \geq n$. But n is the largest positive integer, so $n^2 \leq n$. We conclude that $n^2 = n$, so since n is not zero, we have $n = 1$.

   (b) Proposition: For any integer $a > 0$, there exists a unique integer $b > 0$ such that $a = b^2$.
   Proof: Suppose that there exist two values b_1 and b_2 such that $b_1^2 = a$ and $b_2^2 = a$ with $b_1 > 0$ and $b_2 > 0$. Subtracting yields $b_1^2 - b_2^2 = 0$ so that $(b_1 - b_2)(b_1 + b_2) = 0$ so that $b_1 = b_2$ or $b_1 = -b_2$. But since $b_1 > 0$ and $b_2 > 0$ we cannot have $b_1 = -b_2$. Therefore $b_1 = b_2$ which means there exists a unique integer b such that $b > 0$ and $a = b^2$, as claimed.

   (c) Proposition: If $a_1 = 1$ and $a_{n+1} = 2a_n - 1$ for all $n \geq 1$, then $a_n = 2^n + 1$ for all n.
   Proof: We use induction on n. The base case $n = 1$ is trivial. For the inductive step, suppose $a_n = 2^n + 1$. Then $a_{n+1} = 2a_n - 1 = 2(2^n + 1) - 1 = 2^{n+1} + 1$ as required.

   (d) Proposition: If $a_1 = 2$, and $a_{n+1} = 4a_n - 4a_{n-1}$ for all $n \geq 1$, then $a_n = 2^n$ for all n.
   Proof: We use strong induction on n. The base case $n = 1$ follows since $a_1 = 2 = 2^1$. For the inductive step, suppose $a_k = 2^k$ for all $k \leq n$. Then $a_{k+1} = 4a_n - 4a_{n-1} = 4 \cdot 2^n - 4 \cdot 2^{n-1} = 4 \cdot 2^n - 2 \cdot 2^n = 2 \cdot 2^n = 2^{n+1}$ as required.

   (e) Proposition: We have $2^n = 1$ for every nonnegative integer n.
   Proof: We use strong induction on n. The base case $n = 0$ holds since $2^0 = 1$. For the inductive step suppose that $2^k = 1$ for all $k \leq n$. Then applying the inductive hypothesis yields $2^{n+1} = 2^n 2^n / 2^{n-1} = 1 \cdot 1/1 = 1$, as desired.

   (f) Proposition: All horses are the same color.
   Proof: We use induction on n, the number of horses. The base case $n = 1$ is trivial because any 1 horse is the same color as itself. For the inductive step, suppose that any $n + 1$ horses are the same color. Ignoring the last horse yields means that we need to show that n horses are the same color, which is true by the induction hypothesis. Therefore the result holds by induction.

   (g) Proposition: For every positive integer n, $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$.
   Proof: We use induction on n. The base case $n = 1$ follows because $1 = \frac{1}{2}(1)(2)$. To show the inductive step, we want $1 + 2 + 3 + \cdots + n + (n + 1) = \frac{1}{2}(n + 1)(n + 2)$. Subtracting $n + 1$ from both sides yields $1 + 2 + 3 + \cdots + n = \frac{1}{2}(n + 1)(n + 2) - (n + 1) = \frac{1}{2}n(n + 1)$ which is true by the induction hypothesis. Therefore the result holds by induction.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Prove the following properties of divisibility:

   (a) If a, b are integers then $a|b$ if and only if $(-a)|b$.

   (b) If a, b, c, x, y are integers with $a|b$ and $a|c$, show that $a|(xb + yc)$.

   (c) If a, b, m are integers with $m \neq 0$, show that $a|b$ if and only if $(ma)|(mb)$.

   (d) If a, b are integers with $a|b$ and $b|a$ then $a = b$ or $a = -b$. [Hint: Use the fact that the only divisors of 1 are 1 and -1. Be careful when $a = 0$ or $b = 0$.]

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4. Show the following:

   (a) For all positive integers n, show that the sum $1^2 + 2^2 + 3^2 + \cdots + n^2$ equals $\dfrac{n(n+1)(2n+1)}{6}$.

   (b) For all positive integers n, show that the sum $3^0 + 3^1 + 3^2 + \cdots + 3^n$ equals $\dfrac{3^{n+1} - 1}{2}$.

   (c) The sequence $\{d_n\}_{n \geq 1}$ is defined recursively by $d_1 = 3$, $d_2 = 9$, and for all $n \geq 3$, $d_n = 2d_{n-1} + 3d_{n-2}$. Prove that $d_n = 3^n$ for all positive integers n.

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5. Recall that an integer n is <u>even</u> if $n = 2a$ for some integer a, while an integer n is <u>odd</u> if $n = 2b + 1$ for some integer b. As noted in class, as a consequence of the division algorithm, every integer is either even or odd, and no integer is both.

   (a) Show that the sum of two even or two odd integers is even, and that the sum of an even integer and an odd integer is odd.

   (b) Show that the product of an even integer with any integer is even, and the product of two odd integers is odd.

   (c) Show that if n is even then n^2 is even, and if n is odd then n^2 is odd.

   (d) Deduce that n^2 is even if and only if n is even. [Hint: What is the contrapositive of "if n is odd then n^2 is odd"?]

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6. The goal of this problem is to prove the Binomial Theorem. First, we define the factorial function as $n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$, so that for example $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and so forth: in general, $n! = n \cdot (n-1)!$ for $n \geq 1$. Now define the <u>binomial coefficient</u> $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ for integers $0 \leq k \leq n$, and note that $\binom{n}{0} = \binom{n}{n} = 1$ for every n.

   (a) Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for every $0 \leq k \leq n$. Conclude in particular that $\binom{n}{k}$ is always an integer.

   (b) Suppose that x and y are arbitrary real numbers. Prove the Binomial Theorem: that $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for any positive integer n. [Hint: Induct on n.]

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7. Suppose that $\mathcal{A} = \{A_i : i \in I\}$ and $\mathcal{B} = \{B_j : j \in J\}$ are two families of sets indexed by I and J respectively.

   (a) Prove that for each $s \in I$ it is true that $\bigcap_{i \in I} A_i \subseteq A_s$.

   (b) Prove that for each $t \in J$ it is true that $B_t \subseteq \bigcup_{j \in J} B_j$.

   (c) Prove that if $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ then $\bigcap_{i \in I} A_i \subseteq \bigcup_{j \in J} B_j$. [Hint: Suppose $S \in \mathcal{A} \cap \mathcal{B}$ and apply (a) and (b) to it.]
