1. With universe $\mathbb{Z}$, take $P(x,y)$ to be the statement "$y = 2x$".

   (a) There are 8 possible ways, listed below, of quantifying both $x$ and $y$ in the statement $P(x,y)$. For each statement, translate it into words and then find its truth value.

      i. $\forall x \forall y,\ P(x,y)$: this means "For all $x$, for all $y$, $y = 2x$". This statement is $\boxed{\text{false}}$ because there exist (many!) choices of $x$ and $y$ where $y = 2x$ is false, such as $x = 1$ and $y = 1$.

      ii. $\forall x \exists y,\ P(x,y)$: this means "For all $x$ there exists a $y$ such that $y = 2x$". This statement is $\boxed{\text{true}}$ because no matter what integer $x$ is, we can in fact take $y = 2x$.

      iii. $\exists x \forall y,\ P(x,y)$: this means "There exists an $x$ such that for all $y$, $y = 2x$". This statement is $\boxed{\text{false}}$ because for any value of $x$, there is (at most!) one value of $y$ with $y = 2x$.

      iv. $\exists x \exists y,\ P(x,y)$: this means "There exists an $x$ such that there exists a $y$ with $y = 2x$". This statement is $\boxed{\text{true}}$ because there are (many!) choices of $x$ and $y$ where $y = 2x$, such as $x = y = 0$.

      v. $\forall y \forall x,\ P(x,y)$: this means "For all $y$, for all $x$, $y = 2x$". This statement is $\boxed{\text{false}}$ because there exist (many!) choices of $x$ and $y$ where $y = 2x$ is false, such as $x = 1$ and $y = 1$.

      vi. $\forall y \exists x,\ P(x,y)$: this means "For all $y$ there exists an $x$ such that $y = 2x$". This statement is $\boxed{\text{false}}$ because if we take $y = 1$, there is no integer $x$ with $y = 2x$.

      vii. $\exists y \forall x,\ P(x,y)$: this means "There exists a $y$ such that for all $x$, $y = 2x$". This statement is $\boxed{\text{false}}$ because no matter what $y$ is, we could always pick $x$ not equal to $y/2$.

      viii. $\exists y \exists x,\ P(x,y)$: this means "There exists a $y$ such that there exists an $x$ with $y = 2x$". This statement is $\boxed{\text{true}}$ because there are (many!) choices of $x$ and $y$ where $y = 2x$ is true, such as $x = y = 0$.

   (b) Of the eight quantified statements in part (a), two pairs will always be logically equivalent for any statement $P(x,y)$. Identify these pairs.

   - As we discussed, we may reverse the order of identical quantifiers without changing the truth value.
   - Therefore, statements (i) $\forall x \forall y,\ P(x,y)$ and (v) $\forall y \forall x,\ P(x,y)$ will always be equivalent, as will statements (iv) $\exists x \exists y,\ P(x,y)$ and (viii) $\exists y \exists x,\ P(x,y)$.

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2. Negate each given statement and then rewrite the result as an equivalent positive statement. (All quantifiers should appear ahead of any negation operators.)

   (a) $\exists x,\ x^2 = 2$.

   - The negation is $\neg[\exists x,\ x^2 = 2] = \forall x, \neg(x^2 = 2) = \boxed{\forall x,\ x^2 \neq 2}$.

   (b) $\exists x \exists y,\ x + y \neq 5$.

   - The negation is $\neg[\exists x \exists y,\ x + y \neq 5] = \forall x\, \neg[\exists y,\ x + y \neq 5] = \forall x \forall y,\ \neg[x + y \neq 5] = \boxed{\forall x \forall y,\ x + y = 5}$.

   (c) $\forall x \exists y \exists z,\ x \cdot y + z > 2$.

   - The negation is $\neg[\forall x \exists y \exists z,\ x \cdot y + z > 2] = \boxed{\exists x \forall y \forall z,\ x \cdot y + z \leq 2}$.

   (d) $\forall a \in A\, \exists b \in B,\ (a \in C \wedge b \in C)$.

   - The negation is $\exists a \in A,\ \forall b \in B,\ \neg(a \in C \wedge b \in C) = \boxed{\exists a \in A,\ \forall b \in B,\ (a \notin C \vee b \notin C)}$.

   (e) There exists an $x$ such that for all $y$, it is true that $y^2 > x$.

   - Symbolically, this is $\exists x \forall y,\ y^2 > x$. The negation is $\neg[\exists x \forall y,\ y^2 > x] = \boxed{\forall x \exists y,\ y^2 \leq x}$: for all $x$ there exists $y$ such that $y^2 \leq x$.

   (f) For all $n$ there exist $a$, $b$, and $c$ such that $n = a^2 + b^2 + c^2$.

   - Symbolically, this is $\forall n \exists a \exists b \exists c,\ n = a^2 + b^2 + c^2$. The negation is $\neg[\forall n \exists a \exists b \exists c,\ n = a^2 + b^2 + c^2] = \boxed{\exists n \forall a \forall b \forall c,\ n \neq a^2 + b^2 + c^2}$: there exists $n$ such that for all $a$, $b$, and $c$, $n \neq a^2 + b^2 + c^2$.

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3. For each statement, translate it into words and then find its truth value. (Assume that all capital-letter variables refer to sets.)

   (a) $\forall x \in \mathbb{R},\ x^2 > 0$.

   - This means "For any real number $x$, $x^2 > 0$".
   - This statement is $\boxed{\text{false}}$ because there is a counterexample, namely $x = 0$, for which $x^2 > 0$ is false.

   (b) $\exists x \in \mathbb{Z},\ x^2 - 3x + 2 = 0$.

   - This means "There exists an integer $x$ such that $x^2 - 3x + 2 = 0$".
   - This statement is $\boxed{\text{true}}$ because $x = 1$ is such an integer (as is $x = 2$).

   (c) $\forall A \forall B \forall C,\ [x \in A \cap B \cap C] \Rightarrow [x \in A \cap B] \wedge [x \in A \cup C]$.

   - This means "For any $A$, for any $B$, for any $C$, $x \in A \cap B \cap C$ implies $x \in A \cap B$ and $x \in A \cup C$".
   - This statement is $\boxed{\text{true}}$ because if $x \in A \cap B \cap C$ then $x \in A$ and $x \in B$ and $x \in C$, and therefore $x \in A \cap B$ and also $x \in A \cup C$.

   (d) $\forall A \exists x (x \in A)$.

   - This means "For any $A$ there exists $x$ such that $x \in A$".
   - This statement is $\boxed{\text{false}}$ because $A$ could be the empty set, in which case there is no element $x \in A$.

   (e) $\forall A \exists x \exists y,\ (A = \emptyset) \vee [(x \in A) \wedge (y \in A)]$.

   - This means "For any $A$ there exists $x$ such that there exists $y$ such that either ($A$ is the empty set) or $x \in A$ and $y \in A$".
   - This statement is $\boxed{\text{true}}$: if $A$ is the empty set then the conclusion holds, and if $A$ is not the empty set, then by definition it contains at least one element $e$. We can then take $x = e$ and $y = e$, and the conclusion again holds. (Note that there is no requirement or expectation that $x$ and $y$ must be different elements.)

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4. Each item below contains a proposition (which may be true or may be false) and an *incorrect* proof of the proposition. Identify at least one mistake in each claimed proof:

   (a) <u>Proposition</u>: For all real numbers $x$, it is true that $x^2 \geq 2$.
   <u>Proof</u>: Suppose by way of contradiction that the desired result is false. Then for all real numbers $x$, it is true that $x^2 < 2$. But this statement is incorrect, because taking $x = 3$ yields the false statement $9 < 2$. This is a contradiction, so it must be true that for all real numbers $x$, $x^2 \geq 2$.

   - The error is that the original statement is not negated correctly: it says "for all real $x$, $x^2 \geq 0$", and the claimed negation in the proposition is "for all real $x$, it is true that $x^2 < 0$".
   - But negation swaps quantifiers: the correct negation is "there exists a real $x$ such that $x^2 < 0$", and this result is not contradicted just by noting that the inequality is false when $x = 3$.

   (b) <u>Proposition</u>: For all real numbers $x$ there exists a real number $y$ such that $x - 2y = y^2 + 1$.
   <u>Proof</u>: Let $x = (y+1)^2$: then $x - 2y = (y+1)^2 - 2y = (y^2 + 2y + 1) - 2y = y^2 + 1$, as required.

   - The error is that the order of the quantifiers does not allow us to select the value of $x$ in terms of $y$. The variables are quantified with $x$ first, then $y$ second, so we can only define the value of $y$ in terms of $x$.
   - In this case, for some values of $x$ there need not exist any possible value of $y$: for example, if $x = -5$ then we would require $(y+1)^2 = -5$, which is impossible.

   (c) <u>Proposition</u>: There exists a real number $y$ such that for all real numbers $x$, it is true that $x - 2y = y^2 + 1$.
   <u>Proof</u>: Let $x = (y+1)^2$: then $x - 2y = (y+1)^2 - 2y = (y^2 + 2y + 1) - 2y = y^2 + 1$, as required.

   - The error is that the quantifiers do not allow us to select the value $x$ at all: for a specific $y$, the statement must hold for all possible values of $x$, whereas the proof only shows that a single value of $x$ actually works.

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5. The goal of this problem is to examine the quantifier "there exists a unique", written as $\exists!$. Thus, for example, the statement "there exists a unique $x$ such that $x^2 = 2$" would be written $\exists! x,\ x^2 = 2$. The meaning of this quantifier is that there exists an element $x$ satisfying the hypotheses, and that there is exactly one such $x$.

   (a) Identify the truth values of the following statements:

   i. $\exists! n \in \mathbb{Z},\ n^2 = 2$.
      - This statement means "There exists a unique integer $n$ such that $n^2 = 2$."
      - This statement is $\boxed{\text{false}}$, because there is no such integer $n$: neither $\sqrt{2}$ nor $-\sqrt{2}$ is an integer.

   ii. $\exists! n \in \mathbb{Z},\ n^2 = 4$.
      - This statement means "There exists a unique integer $n$ such that $n^2 = 4$."
      - This statement is $\boxed{\text{false}}$, because there are two such integers $n$: namely, $n = 2$ and $n = -2$.

   iii. $\exists! n \in \mathbb{Z},\ n^2 = 0$.
      - This statement means "There exists a unique integer $n$ such that $n^2 = 0$."
      - This statement is $\boxed{\text{true}}$, because the only integer with $n^2 = 0$ is $n = 0$.

   iv. $\forall x \in \mathbb{R}\ \exists! y \in \mathbb{R},\ x = y^2$.
      - This statement means "For all real $x$, there exists a unique real $y$ such that $x = y^2$".
      - This statement is $\boxed{\text{false}}$, because (e.g.,) for $x = 4$ there are two such $y$, namely $y = 2, -2$.

   v. $\forall x \in \mathbb{R}\ \exists! y \in \mathbb{R},\ x = y^2$.
      - This item accidentally duplicated part (iv).

   vi. $\exists! x \in \mathbb{R}\ \exists! y \in \mathbb{R},\ x^2 + y^2 = 0$.
      - This statement means "There exists a unique real $x$ such that there exists a unique real $y$ such that $x^2 + y^2 = 0$".
      - This statement is $\boxed{\text{true}}$, because there is only one choice of real numbers $x, y$ with $x^2 + y^2 = 0$, namely, $x = 0$ and $y = 0$.

   (b) It may seem that $\exists!$ is a new quantifier, but in fact, it can be expressed in terms of $\exists$ and $\forall$. Briefly explain why $\exists! x \in A,\ P(x)$ is logically equivalent to $\exists x \in A,\ P(x) \wedge [\forall y \in A,\ P(y) \Rightarrow (y = x)]$ for any proposition $P(x)$. (Your explanation does not have to be fully rigorous.)

   - The second statement means "There exists $x \in A$ such that $P(x)$ is true and, for any $y \in A$, if $P(y)$ is true then $y = x$".
   - By unpacking this statement, the first part says that there is at least one $x \in A$ such that $P(x)$ is true, and the second part says that if $y \in A$ is any element of $A$ where $P(y)$ is true, then $y = x$.
   - In other words, the second part says that $x$ must be the only element of $A$ for which $P$ is true. Together with the first part, this says that there exists a unique element of $A$ (namely, $x$) for which $P(x)$ is true, which is the same as saying $\exists! x \in A,\ P(x)$.
   - Another approach: to say there exists a unique $x$ such that $P(x)$ is true is to say there exists an $x$ for which $P(x)$ is true, and there does not exist a $y$ with $y \neq x$ such that $P(y)$ is true. This is the statement $\exists x \in A,\ P(x) \wedge \neg[\exists y \in A,\ P(y) \wedge (y \neq x)]$, which upon moving the negation to the right is seen to be equivalent to the statement given above.

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6. Many of our proofs involving sets have implicitly used quantifiers. The goal of this problem is to make some of these ideas more explicit by analyzing the proof (given in class) that if $A$ and $B$ are any sets then $A \cap B = A$ is logically equivalent to $A \subseteq B$.

   (a) If $P$ and $Q$ are propositions, show that $P \Rightarrow (P \wedge Q)$ is logically equivalent to $P \Rightarrow Q$.

   - By using a truth table we can see that the column for $P \Rightarrow (P \wedge Q)$ is identical to the one for $P \Rightarrow Q$, so they are logically equivalent:

| $P$ | $Q$ | $P \Rightarrow Q$ | $P \wedge Q$ | $P \Rightarrow (P \wedge Q)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | F |
| F | T | T | F | T |
| F | F | T | F | T |

   - Alternatively, we can work it out using Boolean algebra: we have $P \Rightarrow (P \wedge Q)$ iff $\neg P \vee (P \wedge Q)$ iff $(\neg P \vee P) \wedge (\neg P \wedge Q)$ iff True $\wedge (\neg P \wedge Q)$ iff $\neg P \wedge Q$ iff $P \Rightarrow Q$, as desired.

   (b) If $A$ and $B$ are sets, show that $\forall x, (x \in A) \Rightarrow [(x \in A) \wedge (x \in B)]$ is logically equivalent to $\forall x, (x \in A) \Rightarrow (x \in B)$. [Hint: Use (a).]

   - By setting $P = (x \in A)$ and $Q = (x \in B)$ in the result from (a), we see that $(x \in A) \Rightarrow [(x \in A) \wedge (x \in B)]$ is logically equivalent to $(x \in A) \Rightarrow (x \in B)$.
   - This is true regardless of the value of $x$, which is to say, $\forall x, (x \in A) \Rightarrow [(x \in A) \wedge (x \in B)]$ is logically equivalent to $\forall x, (x \in A) \Rightarrow (x \in B)$.

   (c) Explain why (b) says that $A \subseteq A \cap B$ is equivalent to $A \subseteq B$.

   - The statement $\forall x, (x \in A) \Rightarrow (x \in B)$ is merely the definition of $A \subseteq B$.
   - So translating the result from (b) into the language of subset containment yields the claimed fact: $A \subseteq A \cap B$ is logically equivalent to $A \subseteq B$.

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7. The goal of this problem is to examine an example of how quantifiers can affect propositional logic.

   (a) If $P$ and $Q$ are propositions, show that at least one of the two statements $P \Rightarrow Q$ and $Q \Rightarrow P$ is true.

   - By using a truth table we can see that at least one of $P \Rightarrow Q$ and $Q \Rightarrow P$ is true in all cases:

| $P$ | $Q$ | $P \Rightarrow Q$ | $Q \Rightarrow P$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | T | F |
| F | F | T | T |

   (b) Show that neither the statement "if $x = 1$ then $x = 2$" nor its converse are true.

   - Clearly, if $x = 1$ then $x \neq 2$, so "if $x = 1$ then $x = 2$" is false.
   - Likewise, if $x = 2$ then $x \neq 1$, so "if $x = 2$ then $x = 1$" is also false.

   Part (b) would seem to provide a counterexample to part (a), but of course it does not. The reason is that the variable $x$ in part (b) is implicitly quantified at the start as $\forall x$, and the presence and placement of the quantifier affects the logic of the statement. To examine further, suppose $P(x)$ and $Q(x)$ are propositions.

   (c) Show that $\forall x, [P(x) \Rightarrow Q(x)] \vee [Q(x) \Rightarrow P(x)]$ is true.

   - By part (a), we know that at least one of the conditionals $P(x) \Rightarrow Q(x)$ and $Q(x) \Rightarrow P(x)$ is true, so the statement $[P(x) \Rightarrow Q(x)] \vee [Q(x) \Rightarrow P(x)]$ is always true.
   - Therefore, the universal statement $\forall x, [P(x) \Rightarrow Q(x)] \vee [Q(x) \Rightarrow P(x)]$ is true.

   (d) Is the statement $[\forall x, P(x) \Rightarrow Q(x)] \vee [\forall x, Q(x) \Rightarrow P(x)]$ necessarily true?

   - The statement $[\forall x, P(x) \Rightarrow Q(x)] \vee [\forall x, Q(x) \Rightarrow P(x)]$ means that for any $x$ it is true that $P(x) \Rightarrow Q(x)$, or that for any $x$ it is true that $Q(x) \Rightarrow P(x)$.
   - This is not always a true statement, since it may not be the case that either statement always implies the other. The counterexample in part (b) works here: taking $P(x)$ to be $x = 1$ and $Q(x)$ to be $x = 2$, we see that the compound statement is false.

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8. Suppose $A$ and $B$ are sets. The goal of this problem is to study the question of when $A \times B = B \times A$.

Proposition: $A \times B = B \times A$ if and only if $A = B$.
Proof: If $A = B$, then clearly $A \times B = A \times A = B \times A$. Now suppose $A \times B = B \times A$, and let $a \in A$ and $b \in B$ be arbitrary elements of $A$ and $B$ respectively. Then by definition, $(a, b) \in A \times B$, and so by hypothesis, $(a, b) \in B \times A$. This means $a \in B$ and $b \in A$. Since $a \in A$ and $b \in B$ are arbitrary, the fact that $a \in B$ implies $A \subseteq B$, and the fact that $b \in A$ implies $B \subseteq A$. We conclude that $A = B$, as required.

(a) Consider the proposition and proof given above. Show that the proposition is incorrect by explaining why taking $A = \emptyset$ and $B = \{1, 2\}$ yields a counterexample.

- Observe that if $A = \emptyset$ and $B = \{1, 2\}$, then $A \times B = \emptyset$ and $B \times A = \emptyset$, so in particular $A \times B = B \times A$.
- This yields a counterexample because $A \times B = B \times A$ but $A$ and $B$ are not the same set.

(b) Part (a) shows that the proposition stated above is incorrect, so the proof must contain a logical error. Identify what the error is, and why it causes the proof to be incorrect. [Hint: The counterexample from part (a) is clearly relevant.]

- The error in the proof is the statement "let $a \in A$ and $b \in B$ be arbitrary elements of $A$ and $B$ respectively".
- We can only select an element of $A$ and an element of $B$ if $A$ and $B$ are both nonempty, so in the event that $A = \emptyset$ or $B = \emptyset$, the proof is erroneous because of this statement.

(c) Give a corrected version of the proposition, and then give a correct proof. [Hint: Your corrected proposition should start with "$A \times B = B \times A$ if and only if $A = B$ or..." and the proof should also make sure to address the error you identified in part (b).]

- The correct statement is as follows:
  Proposition: $A \times B = B \times A$ if and only if $A = B$ or $A = \emptyset$ or $B = \emptyset$.
- Here is a proof:
- First, if $A = \emptyset$, then $A \times B = \emptyset = B \times A$ so the result holds. Likewise, if $B = \emptyset$, then $A \times B = \emptyset = B \times A$ so the result holds as well.
- Now suppose that $A$ and $B$ are nonempty. At this point, we can simply quote the proof given above, because it is correct as long as $A$ and $B$ are nonempty.
- Remark: A common error was not to address the cases where $A = \emptyset$ or $B = \emptyset$ in the corrected proof. If this is not done before the statement "Let $a \in A$ and $b \in B$ be arbitrary elements of $A$ and $B$ respectively", then the new proof is still wrong for the reasons explained in (b) above.

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