E. Dummit's Math 1465 ~ Intensive Mathematical Reasoning, Fall 2024 ~ Homework 2, due Tue Sep 24th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. With universe $\mathbb{Z}$, take $P(x, y)$ to be the statement "$y = 2x$".

   (a) There are 8 possible ways, listed below, of quantifying both x and y in the statement $P(x, y)$. For each statement, translate it into words and then find its truth value.

   i. $\forall x \forall y,\ P(x, y)$
   ii. $\forall x \exists y,\ P(x, y)$
   iii. $\exists x \forall y,\ P(x, y)$
   iv. $\exists x \exists y,\ P(x, y)$
   v. $\forall y \forall x,\ P(x, y)$
   vi. $\forall y \exists x,\ P(x, y)$
   vii. $\exists y \forall x,\ P(x, y)$
   viii. $\exists y \exists x,\ P(x, y)$

   (b) Of the eight quantified statements in part (a), two pairs will always be logically equivalent for any statement $P(x, y)$. Identify these pairs.

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2. Negate each given statement and then rewrite the result as an equivalent positive statement. (All quantifiers should appear ahead of any negation operators.)

   (a) $\exists x,\ x^2 = 2$.
   (b) $\exists x \exists y,\ x + y \neq 5$.
   (c) $\forall x \exists y \exists z,\ x \cdot y + z > 2$.
   (d) $\forall a \in A\, \exists b \in B,\ (a \in C \wedge b \in C)$.
   (e) There exists an x such that for all y, it is true that $y^2 > x$.
   (f) For all n there exist a, b, and c such that $n = a^2 + b^2 + c^2$.

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3. For each statement, translate it into words and then find its truth value. (Assume that all capital-letter variables refer to sets.)

   (a) $\forall x \in \mathbb{R},\ x^2 > 0$.
   (b) $\exists x \in \mathbb{Z},\ x^2 - 3x + 2 = 0$.
   (c) $\forall A \forall B \forall C,\ [x \in A \cap B \cap C] \Rightarrow [x \in A \cap B] \wedge [x \in A \cup C]$.
   (d) $\forall A \exists x (x \in A)$.
   (e) $\forall A \exists x \exists y,\ (A = \emptyset) \vee [(x \in A) \wedge (y \in A)]$.

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4. Each item below contains a proposition (which may be true or may be false) and an *incorrect* proof of the proposition. Identify at least one mistake in each claimed proof:

   (a) Proposition: For all real numbers x, it is true that $x^2 \geq 2$.
   Proof: Suppose by way of contradiction that the desired result is false. Then for all real numbers x, it is true that $x^2 < 2$. But this is statement is incorrect, because taking $x = 3$ yields the false statement $9 < 2$. This is a contradiction, so it must be true that for all real numbers x, $x^2 \geq 2$.

   (b) Proposition: For all real numbers x there exists a real number y such that $x - 2y = y^2 + 1$.
   Proof: Let $x = (y + 1)^2$: then $x - 2y = (y + 1)^2 - 2y = (y^2 + 2y + 1) - 2y = y^2 + 1$, as required.

   (c) Proposition: There exists a real number y such that for all real numbers x, it is true that $x - 2y = y^2 + 1$.
   Proof: Let $x = (y + 1)^2$: then $x - 2y = (y + 1)^2 - 2y = (y^2 + 2y + 1) - 2y = y^2 + 1$, as required.

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5. The goal of this problem is to examine the quantifier "there exists a unique", written as $\exists!$. Thus, for example, the statement "there exists a unique x such that $x^2 = 2$" would be written $\exists! x,\ x^2 = 2$. The meaning of this quantifier is that there exists an element x satisfying the hypotheses, and that there is exactly one such x.

   (a) Identify the truth values of the following statements:

   i. $\exists! n \in \mathbb{Z},\ n^2 = 2$.
   ii. $\exists! n \in \mathbb{Z},\ n^2 = 4$.
   iii. $\exists! n \in \mathbb{Z},\ n^2 = 0$.
   iv. $\forall x \in \mathbb{R}\ \exists! y \in \mathbb{R},\ x = y^2$.
   v. $\forall x \in \mathbb{R}\ \exists! y \in \mathbb{R},\ x = y^2$.
   vi. $\exists! x \in \mathbb{R}\ \exists! y \in \mathbb{R},\ x^2 + y^2 = 0$.

   (b) It may seem that $\exists!$ is a new quantifier, but in fact, it can be expressed in terms of $\exists$ and $\forall$. Briefly explain why $\exists! x \in A,\ P(x)$ is logically equivalent to $\exists x \in A,\ P(x) \wedge [\forall y \in A,\ P(y) \Rightarrow (y = x)]$ for any proposition $P(x)$. (Your explanation does not have to be fully rigorous.)

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

6. Many of our proofs involving sets have implicitly used quantifiers. The goal of this problem is to make some of these ideas more explicit by analyzing the proof (given in class) that if A and B are any sets then $A \cap B = A$ is logically equivalent to $A \subseteq B$.

   (a) If P and Q are propositions, show that $P \Rightarrow (P \wedge Q)$ is logically equivalent to $P \Rightarrow Q$.

   (b) If A and B are sets, show that $\forall x, (x \in A) \Rightarrow [(x \in A) \wedge (x \in B)]$ is logically equivalent to $\forall x, (x \in A) \Rightarrow (x \in B)$. [Hint: Use (a).]

   (c) Explain why (b) says that $A \subseteq A \cap B$ is equivalent to $A \subseteq B$.

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7. The goal of this problem is to examine an example of how quantifiers can affect propositional logic.

   (a) If P and Q are propositions, show that at least one of the two statements $P \Rightarrow Q$ and $Q \Rightarrow P$ is true.

   (b) Show that neither the statement "if $x = 1$ then $x = 2$" nor its converse are true.

   Part (b) would seem to provide a counterexample to part (a), but of course it does not. The reason is that the variable x in part (b) is implicitly quantified at the start as $\forall x$, and the presence and placement of the quantifier affects the logic of the statement. To examine further, suppose $P(x)$ and $Q(x)$ are propositions.

   (c) Show that $\forall x, [P(x) \Rightarrow Q(x)] \vee [Q(x) \Rightarrow P(x)]$ is true.

   (d) Is the statement $[\forall x, P(x) \Rightarrow Q(x)] \vee [\forall x, Q(x) \Rightarrow P(x)]$ necessarily true?

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8. Suppose A and B are sets. The goal of this problem is to study the question of when $A \times B = B \times A$.

   Proposition: $A \times B = B \times A$ if and only if $A = B$.
   Proof: If $A = B$, then clearly $A \times B = A \times A = B \times A$. Now suppose $A \times B = B \times A$, and let $a \in A$ and $b \in B$ be arbitrary elements of A and B respectively. Then by definition, $(a, b) \in A \times B$, and so by hypothesis, $(a, b) \in B \times A$. This means $a \in B$ and $b \in A$. Since $a \in A$ and $b \in B$ are arbitrary, the fact that $a \in B$ implies $A \subseteq B$, and the fact that $b \in A$ implies $B \subseteq A$. We conclude that $A = B$, as required.

   (a) Consider the proposition and proof given above. Show that the proposition is incorrect by explaining why taking $A = \emptyset$ and $B = \{1, 2\}$ yields a counterexample.

   (b) Part (a) shows that the proposition stated above is incorrect, so the proof must contain a logical error. Identify what the error is, and why it causes the proof to be incorrect. [Hint: The counterexample from part (a) is clearly relevant.]

   (c) Give a corrected version of the proposition, and then give a correct proof. [Hint: Your corrected proposition should start with "$A \times B = B \times A$ if and only if $A = B$ or..." and the proof should also make sure to address the error you identified in part (b).]
