1. Suppose $A = \{1,3,9\}$, $B = \{6,7,8,9\}$, and $C = \{2,3,5,7\}$, with universal set $U = \{1,2,3,4,5,6,7,8,9\}$. Calculate the following:

   (a) $A \cap B$: we have $A \cap B = \boxed{\{9\}}$.

   (b) $A \cup C$: we have $A \cup C = \boxed{\{1,2,3,5,7,9\}}$.

   (c) $(A \cup B) \cup C$: we have $(A \cup B) \cup C = \{1,3,6,7,8,9\} \cap \{2,3,5,7\} = \boxed{\{1,2,3,5,6,7,8,9\}}$.

   (d) $A \cup (B \cup C)$: we have $A \cup (B \cup C) = \{1,3,9\} \cap \{2,3,5,6,7,8,9\} = \boxed{\{1,2,3,5,6,7,8,9\}}$.

   (e) $A \cap (B \cup C)$: we have $A \cap (B \cup C) = \{1,3,9\} \cap \{2,3,5,6,7,8,9\} = \boxed{\{3,9\}}$

   (f) $(A \cap B) \cup C$: we have $(A \cap B) \cup C = \{9\} \cup \{2,3,5,7\} = \boxed{\{2,3,5,7,9\}}$.

   (g) $B^c \cap C^c$: we have $B^c = \{1,2,3,4,5\}$ and $C^c = \{1,4,6,8,9\}$ so $B^c \cap C^c = \boxed{\{1,4\}}$.

   (h) $(A^c \cup B) \cap (B^c \cup C)$: we have $A^c = \{2,4,6,7,8\}$ so $A^c \cup B = \{2,4,6,7,8,9\}$, $B^c \cup C = \{1,2,3,4,5,7\}$, and then $(A^c \cup B) \cap (B^c \cup C) = \boxed{\{2,4,7\}}$.

   (i) $A \times B$: we have $A \times B = \boxed{\{(1,6),(1,7),(1,8),(1,9),(3,6),(3,7),(3,8),(3,9),(9,6),(9,7),(9,8),(9,9)\}}$.

   (j) $B \times A$: we have $B \times A = \boxed{\{(6,1),(7,1),(8,1),(9,1),(6,3),(7,3),(8,3),(9,3),(6,9),(7,9),(8,9),(9,9)\}}$.

   (k) $(A \times B) \cap (B \times A)$: from the above we see $(A \times B) \cap (B \times A) = \boxed{\{(9,9)\}}$.

   (l) $(A \cap B) \times (B \cup C)$: this is $\{9\} \times \{2,3,5,6,7,8,9\} = \boxed{\{(9,2),(9,3),(9,5),(9,6),(9,7),(9,8),(9,9)\}}$.

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2. Suppose $A = \{1,2,\{1\},\emptyset\}$. Identify each of the statements below as true or as false, and give a brief (1-sentence) explanation why.

   (a) $\emptyset \in A$. $\boxed{\text{True}}$: the element $\emptyset$, the empty set, is an element of A.

   (b) $\emptyset \subseteq A$. $\boxed{\text{True}}$: the empty set is a subset of any set.

   (c) $\{1\} \in A$. $\boxed{\text{True}}$: the element $\{1\}$, consisting of the set containing the number 1, is an element of A.

   (d) $\{1\} \subseteq A$. $\boxed{\text{True}}$: the set $\{1\}$, consisting of the set containing the number 1, is a subset of A because $1 \in A$.

   (e) $\{1,2\} \in A$. $\boxed{\text{False}}$: the set $\{1,2\}$, consisting of the set containing the numbers 1 and 2, is not an element of A.

   (f) $\{1,2\} \subseteq A$. $\boxed{\text{True}}$: the set $\{1,2\}$, consisting of the set containing the numbers 1 and 2, is a subset of A because $1 \in A$ and $2 \in A$.

   (g) $\{\{1\}\} \in A$. $\boxed{\text{False}}$: the set $\{\{1\}\}$, consisting of the set whose single element is the set $\{1\}$, is not an element of A.

   (h) $\{\{1\}\} \subseteq A$. $\boxed{\text{True}}$: the set $\{\{1\}\}$, consisting of the set whose single element is the set $\{1\}$, is an element of A because $\{1\} \in A$.

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3. Each item below contains a proposition (which may be true or may be false) and an *incorrect* proof of the proposition. Identify at least one mistake in each claimed proof:

   (a) Proposition: If A and B are sets with $A \cup B = \{1,2,3\}$ then $A \neq \{1\}$ and $B \neq \{2\}$.
   Proof: Suppose by way of contradiction that the desired result is false. Then $A = \{1\}$ and $B = \{2\}$, but then $A \cup B = \{1,2\}$ which contradicts the assumption that $A \cup B = \{1,2,3\}$. This is impossible, so we must have $A \neq \{1\}$ and $B \neq \{2\}$.

   - The error is that the statement "$A \neq \{1\}$ and $B \neq \{2\}$" is not negated correctly. By de Morgan's law $\neg(P \wedge Q) = (\neg P) \vee (\neg Q)$, the negation of "$A \neq \{1\}$ and $B \neq \{2\}$" is "$A = \{1\}$ or $B = \{2\}$", not "$A = \{1\}$ and $B = \{2\}$" as the proof uses.

   (b) Proposition: For any sets A and B inside a universal set U, $(A \cup B) \cap B^c = A$.
   Proof: The union $A \cup B$ consists of all elements in A or in B. Taking the intersection with B^c removes all elements that are in B, leaving only the elements in A. Therefore, $(A \cup B) \cap B^c = A$.

   - The verbal description of the union and intersection are correct: the error is that removing all elements that are in B will also remove elements from A, if A has any elements in common with B, yielding a proper subset of A.
   - An explicit counterexample is $A = \{1,2\}$, $B = \{2,3\}$, $U = \{1,2,3\}$: then $(A \cup B) \cap B^c = \{1,2,3\} \cap \{1\} = \{1\}$, which does not equal A. (The correct statement is in fact $(A \cup B) \cap B^c = A \cap B^c$.)

   (c) Proposition: For any sets A and B inside a universal set U, if $A^c \subseteq B^c$ then $B \subseteq A$.
   Proof: Suppose B is a subset of A, meaning that if $x \in B$ then $x \in A$. Taking the contrapositive of this statement shows that if $x \notin A$ then $x \notin B$, which is the same as saying that if $x \in A^c$ then $x \in B^c$. This shows $A^c \subseteq B^c$, as required.

   - The steps in the proof are all correct, but the proof starts with the assumption that $B \subseteq A$ and proceeds to show that $A^c \subseteq B^c$: that is the converse of the claimed statement "if $A^c \subseteq B^c$ then $B \subseteq A$".

   (d) Proposition: If A, B, C are sets with $A \subseteq C$, $B \subseteq C$, and $x \in A$, then $x \in B$.
   Proof: Suppose by way of contradiction that $x \notin B$. Since $x \in A$ and $A \subseteq C$, this means $x \in C$. Also, since $x \notin B$ and $B \subseteq C$, this means $x \notin C$. These two statements contradict each other, which is impossible. Therefore, we must have $x \in B$.

   - The error is that $x \notin B$ and $B \subseteq C$ does not imply $x \notin C$, as can be seen via the example $x = 2$, $B = \{3\}$, $C = \{2,3\}$. Intuitively, if we are told that x is not an element of the smaller set B, nothing prevents x from being an element of the bigger set C.

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4. Wikipedia has many articles. Some of these articles are lists, such as the article "List of American mathematicians". There are even articles which list other lists, such as the article "List of lists of mathematical topics". Some of these lists of lists contain themselves, such as the article "List of lists of lists".

   (a) Consider the Wikipedia article titled "List of lists that do not contain themselves". Can this article be listed on itself? (Assume, as most students do, that everything on Wikipedia is accurate.)

   - Either possible answer (yes or no) to this question leads to a paradox.
   - If the article is listed on itself, then by definition, it is a list that contains itself. On the other hand, the list is titled "List of lists that do not contain themselves", so it cannot be listed on itself. This is impossible.
   - If the article is not listed on itself, then because it is titled "List of lists that do not contain themselves", it should then be listed on itself. This is also impossible.

   (b) The only true test of a hypothesis is empirical evidence. To this end, what *actually happens* when you visit the Wikipedia article "List of lists that do not contain themselves"?

   - The page redirects to the Wikipedia page for "Russell's Paradox", which is the name of the logical paradox we have just discussed.

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5. In addition to union and intersection, there are a few other set operations that arise from time to time. Two of these are the set difference $A \backslash B = \{x \in A : x \notin B\}$, the set of elements of A not in B, and the symmetric difference $A \Delta B = (A \backslash B) \cup (B \backslash A)$, the elements in either A or B but not both. (Observe that $A \Delta B = B \Delta A$, whence the name *symmetric* difference.)

   (a) If $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6\}$, and $C = \{1, 3, 5\}$, find $A \backslash B$, $B \backslash A$, $A \backslash C$, $C \backslash A$, $B \backslash C$, and $C \backslash B$.

   - We have $A \backslash B = \{1, 3, 5\}$, $B \backslash A = \{6\}$, $A \backslash C = \{2, 4\}$, $C \backslash A = \emptyset$, $B \backslash C = \{2, 4, 6\}$, and $C \backslash B = \{1, 3, 5\}$.

   (b) If $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6\}$, and $C = \{1, 3, 5\}$, find $A \Delta B$, $A \Delta C$, and $B \Delta C$.

   - From the calculations in (a), we see $A \Delta B = \{1, 3, 5, 6\}$, $A \Delta C = \{2, 4\}$, and $B \Delta C = \{1, 2, 3, 4, 5, 6\}$.

   (c) For each of the following statements, decide whether it is true or whether it is false using a Venn diagram or otherwise:

   i. $A \backslash B = A \cap B^c$: $\boxed{\text{True}}$, both $A \backslash B$ and $A \cap B^c$ consist of the region inside A but outside B.

   ii. $A \Delta B = (A \cap B^c) \cup (A^c \cap B)$: $\boxed{\text{True}}$, $(A \cap B^c) \cup (A^c \cap B)$ consists of the region inside exactly one of A and B, and that is also what $A \Delta B$ consists of.

   iii. $(A \backslash B) \cup B = A$: $\boxed{\text{False}}$, per a Venn diagram, $(A \backslash B) \cup B$ is in fact equal to $A \cup B$. A counterexample is $A = \{1, 2\}$ and $B = \{1, 3\}$: then $(A \backslash B) \cup B = \{1, 2, 3\} \neq A$.

   iv. $(A \backslash B) \cup (A \backslash B^c) = A$: $\boxed{\text{True}}$, since $A \backslash B^c = A \cap B$, this reduces to the true statement $(A \cap B^c) \cup (A \cap B) = A$.

   v. $A \Delta B = (A \cup B) \backslash (A \cap B)$: $\boxed{\text{True}}$, $(A \cup B) \backslash (A \cap B)$ consists of the region inside A and outside B along with the region outside A and inside B, which is the same as $A \Delta B$.

   vi. $(A \backslash B) \cap (B \backslash A) = \emptyset$: $\boxed{\text{True}}$, since $(A \backslash B) \cap (B \backslash A) = A \cap B^c \cap B \cap A^c = \emptyset$, these regions do not overlap.

   vii. $(A \Delta B)^c = A^c \backslash B$: $\boxed{\text{False}}$, the left side consists of the two regions $A \cap B$ and $A^c \cap B^c$, while the right side consists only of $A^c \cap B^c$. A counterexample is $A = \{1, 2\}$ and $B = \{1, 3\}$ inside $\{1, 2, 3, 4\}$: then $(A \Delta B)^c = \{1, 4\}$ while $A^c \backslash B = \{4\}$.

   viii. $(A \backslash B) \backslash C = (A \backslash C) \backslash B$: $\boxed{\text{True}}$, both sides consist of the single region inside A but outside both B and C.

   ix. $(A \Delta B) \Delta C = A \Delta (B \Delta C)$: $\boxed{\text{True}}$, both sides consist of the regions inside exactly one of A, B, C along with the region inside all three of them.

   x. $(A \cap B) \Delta C = (A \Delta C) \Delta (A \backslash B)$: $\boxed{\text{True}}$, both sides consist of the four regions $A \cap B \cap C^c$, $A^c \cap B \cap C$, $A \cap B^c \cap C$, and $A^c \cap B^c \cap C$.

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6. A common strategy to show that two sets S and T are equal is to show that $S \subseteq T$ (by establishing that $x \in S$ implies $x \in T$) and also that $T \subseteq S$ (by establishing that $x \in T$ implies $x \in S$). Using this method, or otherwise, and supposing that A, B, and C are any sets contained in a universal set U, provide a rigorous proof of each of the following statements (in particular, you may *not* appeal to Venn diagrams in your solutions and must use *only* formal properties of set containments):

   (a) Prove that $A \cap (A \cap B) = A \cap B$.

   - First suppose $x \in A \cap (A \cap B)$. Then by definition, $x \in A$ and $x \in A \cap B$, so in particular $x \in A \cap B$. Thus, $A \cap (A \cap B) \subseteq A \cap B$.
   - For the other containment, suppose $x \in A \cap B$. Then by definition, $x \in A$ and $x \in B$, so in particular $x \in A$ and $x \in A \cap B$. Thus, $x \in A \cap (A \cap B)$, so $A \cap B \subseteq A \cap (A \cap B)$, so together with the above, we have equality.

(b) Prove that $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.

- First suppose $(x, y) \in (A \times B) \cap (B \times A)$. Then by definition of the intersection, $(x, y) \in A \times B$ and $(x, y) \in B \times A$. By definition of the Cartesian product (twice), we have $x \in A$ and $y \in B$, and also $x \in B$ and $y \in A$. But then since $x \in A$ and $x \in B$ this means $x \in A \cap B$, and also since $y \in A$ and $y \in B$ this means $y \in A \cap B$.
- Therefore, again by definition of the Cartesian product, this means $(x, y) \in (A \cap B) \times (A \cap B)$. Thus, $(A \times B) \cap (B \times A) \subseteq (A \cap B) \times (A \cap B)$.
- For the other containment, suppose $(x, y) \in (A \cap B) \times (A \cap B)$. By definition of the Cartesian product, this means $x \in A \cap B$ and $y \in A \cap B$, which in turn by the definition of intersection means $x \in A$ and $x \in B$ and also $y \in A$ and $y \in B$.
- Therefore, by definition of the Cartesian product since $x \in A$ and $y \in B$ we have $(x, y) \in A \times B$, and also since $x \in B$ and $y \in A$ we have $(x, y) \in B \times A$. Thus, by the definition of intersection, we have $(x, y) \in (A \cap B) \times (B \cap A)$. We conclude $(A \cap B) \times (A \cap B) \subseteq (A \times B) \cap (B \times A)$, and therefore we have equality.

(c) Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

- First suppose $(x, y) \in A \times (B \cup C)$. Then by definition of the Cartesian product, $x \in A$ and $y \in B \cup C$, meaning that $y \in B$ or $y \in C$ by the definition of intersection.
- Hence, again by definition of the Cartesian product, we have $(x, y) \in A \times B$ or $(x, y) \in A \times C$, and therefore $(x, y) \in (A \times B) \cup (A \times C)$ by the definition of intersection. Thus, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.
- For the other containment, suppose $(x, y) \in (A \times B) \cup (A \times C)$. Then by definition, $(x, y) \in A \times B$ or $(x, y) \in A \times C$, so by the definition of Cartesian product, $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$.
- Thus, $x \in A$ and $y \in B \cup C$, so $(x, y) \in A \times (B \cup C)$. We conclude $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$, and therefore we have equality.
- Remark: Common errors included only establishing one implication (e.g., showing that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ without also showing the reverse containment) and skipping steps or not phrasing arguments rigorously.

Remark: Note that some of these parts ask for a proof of a result stated but not explicitly proven in the course notes. As such, you may NOT quote those results in the responses (e.g., your response to part (c) cannot be "This follows immediately from the distributive laws for Cartesian products"), because it would be circular logic.

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7. Suppose A and B are sets. Prove that $A \subseteq B$ if and only if $A \cup B = B$.

- Since the statement is an "if and only if" we must show that $A \subseteq B$ implies $A \cup B = B$, and also that $A \cup B = B$ implies $A \subseteq B$.
- So first suppose $A \subseteq B$. If $x \in A \cup B$, then $x \in A$ or $x \in B$. By the definition of $A \subseteq B$, we know that $x \in A$ implies $x \in B$. Therefore, $x \in A$ or $x \in B$ implies $x \in B$, and thus $A \cup B \subseteq B$. Since $B \subseteq A \cup B$ for any sets A and B, we conclude that $A \cup B = B$.
- Now suppose $A \cup B = B$. Then in particular, $A \cup B \subseteq B$, which means that $x \in A \cup B$ implies $x \in B$. By definition, this says $x \in A$ or $x \in B$ implies $x \in B$, and thus in particular, $x \in A$ implies $x \in B$. This means $A \subseteq B$, as required.
- We have established both implications, so we are done.

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8. In some mathematical arguments, there can arise several different possible cases. In such situations, it can be useful to break the argument apart and analyze each of the possible cases separately, rather than trying to deal with them all at once. Indeed, the idea of breaking into cases is the procedure underlying the use of truth tables: we simply evaluate logical expressions in all possible cases to see whether they agree. Using this idea or otherwise, prove the following:

   (a) If A, B, C are any sets, prove that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

   - Suppose $x \in (A \cap B) \cup (A \cap C)$. Then by definition of the union, $x \in A \cap B$ or $x \in A \cap C$. We see that we have two possible cases: either $x \in A \cap B$ or $x \in A \cap C$.
   - In the case where $x \in A \cap B$ then by definition of the intersection, $x \in A$ and $x \in B$, so in particular since $x \in B$ we see $x \in B \cup C$. Therefore since $x \in A$ and $x \in B \cup C$ we have $x \in A \cap (B \cup C)$ by definition of the intersection.
   - In the case where $x \in A \cap C$ then by definition of the intersection, $x \in A$ and $x \in C$, so in particular since $x \in C$ we see $x \in B \cup C$. Therefore since $x \in A$ and $x \in B \cup C$ we again have $x \in A \cap (B \cup C)$ by definition of the intersection.
   - In both cases, we see that $x \in A \cap (B \cup C)$, and so we conclude that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ as claimed.

   (b) If A, B, C are any sets, prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, and deduce in fact that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   - Suppose $x \in A \cap (B \cup C)$. Then by definition of the intersection, $x \in A$ and also $x \in B \cup C$, meaning that $x \in B$ or $x \in C$. We see that we have two possible cases: either $x \in B$ or $x \in C$.
   - In the case where $x \in B$ then since we also know $x \in A$, by definition of the intersection we see that $x \in A \cap B$. Thus, in particular, we have $x \in (A \cap B) \cup (A \cap C)$.
   - In the case where $x \in C$ then since we also know $x \in A$, by definition of the intersection we see that $x \in A \cap C$. Thus, in particular, we again have $x \in (A \cap B) \cup (A \cap C)$.
   - In both cases, we see that $x \in (A \cap B) \cup (A \cap C)$, and so we conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ as claimed.
   - Combining this result with part (a) yields the desired equality $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   (c) If A, B are any subsets of a universal set U, prove that $(A \cup B)^c = A^c \cap B^c$. [Hint: For any $x \in U$, there are four possible cases: either $x \in A, x \in B$ or $x \in A, x \notin B$ or $x \notin A, x \in B$ or $x \notin A, x \notin B$.]

   - Let $x \in U$. Per the hint we will analyze the four possible cases $x \in A, x \in B$ or $x \in A, x \notin B$ or $x \notin A, x \in B$ or $x \notin A, x \notin B$.
   - If $x \in A$ and $x \in B$, then $x \in A \cup B$ and so $x \notin (A \cup B)^c$. Also, $x \notin A$ so $x \notin (A^c \cap B^c)$.
   - If $x \in A$ and $x \notin B$, then $x \in A \cup B$ and so $x \notin (A \cup B)^c$. Also, $x \notin B$ so $x \notin (A^c \cap B^c)$.
   - If $x \notin A$ and $x \in B$, then $x \in A \cup B$ and so $x \notin (A \cup B)^c$. Also, $x \notin B$ so $x \notin (A^c \cap B^c)$.
   - If $x \notin A$ and $x \notin B$, then $x \notin A \cup B$ and so $x \in (A \cup B)^c$. Also, $x \in A^c$ and $x \in B^c$ so $x \in (A^c \cap B^c)$.
   - From the four cases, we see that $x \in (A \cup B)^c$ only in the situation where $x \notin A$ and $x \notin B$, and this is also the only situation where $x \in A^c \cap B^c$.
   - Since the cases exhaust all possibilities, we conclude that $x \in (A \cup B)^c$ if and only if $x \in A^c \cap B^c$, meaning that $(A \cup B)^c = A^c \cap B^c$ as claimed.
   - Remark: This argument is really just a formalization of a Venn diagram calculation, since the four regions in the Venn diagram correspond to the four possible cases analyzed here.
